Chapter 7

Simple Roots

Now let us return to the general theory, using SU(3) and SU(2) as examples.

Consider a compact semisimple finite dimensional Lie algebra $\mathcal{L}$ (as usual) with a definite Cartan subalgebra and a fixed basis for it, $H_1, H_2, \ldots, H_m$. Any representation can be written with a basis in which the H_i are diagonal, so the m eigenvalues belonging to a given basis vector in the representation is an m component vector $\mu_1, \ldots \mu_m$ which we have already defined as the weight.

We give an ordering to the weights by an alphabetic method, that is, $\mu > \mu'$ if the first nonzeroth element of the vector $\vec{\mu} - \vec{\mu}'$ is greater than zero. $\vec{\mu} = \vec{\mu}'$ only if the vectors are identical.

This definition is clearly arbitrary, because it depends on our choice of basis for $\mathcal{H}$, even on the order in which we list the basis elements. It is nonetheless useful. A weight is **positive** if it is > 0.

The adjoint representation is also subject to this definition, so we have imposed an ordering on the roots[1]. Now all the roots are either positive or negative. The positive roots can be considered raising operators, and the negative ones their conjugate lowering operators.

Note because Georgi has chosen $H_1 = T_3$ and $H_2 = T_8$, instead of the other way around, the raising operators are T_+, V_+, and U_-, not U_+. If he had interchanged H_1 and H_2, T_+, V_+, and U_+ would have been the raising operators. Which order one chooses is not important as long as you stick to it, and as we are following Georgi, we better stick to his.

When we want to know how the algebra acts on a representation vector

[1] We have shown (page 89) that no two generators have the same root.

space, we don't need to investigate how $[A, B]$ operates if we already know how A and B work separately. So if we already know the action of two roots $E_{\vec{\alpha}}$ and $E_{\vec{\beta}}$, we don't need to worry about $\left[E_{\vec{\alpha}}, E_{\vec{\beta}}\right] \propto E_{\vec{\alpha}+\vec{\beta}}$, even if it is a root. We therefore define

A **simple root** is a positive root which cannot be written as a sum of two positive roots.

Lemma: If $\vec{\alpha}$ and $\vec{\beta}$ are unequal simple roots, $\vec{\alpha}-\vec{\beta}$ is not a root, and $\vec{\alpha}\cdot\vec{\beta} \leq 0$.

Proof: If $\vec{\alpha} - \vec{\beta}$ is a root, so is $\vec{\beta} - \vec{\alpha}$, and one of them is positive, say $\vec{\alpha} - \vec{\beta}$. Then $\vec{\alpha}$ is the sum of positive roots $\vec{\alpha} - \vec{\beta}$ and $\vec{\beta}$, and is not simple, which contradicts the hypothesis.

If $\vec{\beta} - \vec{\alpha}$ is not a root, the q for the $\vec{\alpha}$ multiplet starting from $\vec{\beta}$ is zero, and $\dfrac{\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} = -\dfrac{p}{2} \leq 0$.

The set of nonnegative integers p for each ordered pair $\vec{\alpha}$, $\vec{\beta}$ determine not only the angles $\cos^2\theta_{\alpha\beta} = p_{\alpha\beta}p_{\beta\alpha}/4$ but also the relative magnitudes $\beta^2/\alpha^2 = p_{\alpha\beta}/p_{\beta\alpha}$ for all the simple root vectors. And each p can only be 0, 1, 2, or 3. Also either both $p_{\alpha\beta}$ and $p_{\beta\alpha}$ are zero or at least one of the two is 1.

Now if there aren't too many simple roots we should be able to determine everything. We now show that the number of simple roots is equal to the rank m of the group.

First, we show that they are linearly independant. Suppose not, so there exists a relation $\sum_i x_i\alpha^i = 0$ with some real $x_i \neq 0$. Divide the expression into two parts with the coefficients positive or negative.

$$\sum_{i\ni x_i>0} x_i\vec{\alpha}^{\,i} + \sum_{i\ni x_i<0} x_i\vec{\alpha}^{\,i} = 0,$$

or, calling the first piece $y := \sum_{i\ni x_i>0} x_i\vec{\alpha}^{\,i}$ and minus the second $z := \sum_{i\ni x_i<0} |x_i|\vec{\alpha}^{\,i}$, we have $y = z$. Now y and z are both positive linear combinations of positive roots, so each is nonzero, but

$$y^2 = y \cdot z = \sum_{\substack{i\ni x_i>0 \\ j\ni x_j<0}} |x_i||x_j|\vec{\alpha}^{\,i} \cdot \vec{\alpha}^{\,j} \leq 0,$$

because all the included $\vec{\alpha}^{\,i} \cdot \vec{\alpha}^{\,j}$ are ≤ 0. This contradicts $y^2 > 0$ as $y \neq 0$. QED.

Thus the number of positive roots cannot be greater than the dimensionality of the space they live in, which is the rank of the algebra.

Second, we show any positive root ϕ is a sum of one or more simple roots $\vec{\alpha}_i$ with nonnegative integer coefficients $K_{\vec{\alpha}_i}$. Suppose not. Then there is a smallest positive root which cannot be so written, and it is not simple, so it can be written as a sum of two positive roots, each smaller than it, and hence expressible as such.

Third, we show that the simple roots span the whole m dimensional space, and hence there are exactly m of them. For if not, let v be a vector in the m dimensional space perpendicular to all the simple roots. Every root is a linear combination of these (as either it is positive or it is minus a positive root), so $\vec{v}\cdot\vec{\alpha}=0$ for every $E_{\vec{\alpha}}$.

Then $\left[\vec{v}\cdot\vec{H}, E_{\vec{\alpha}}\right] = \vec{v}\cdot\vec{\alpha}=0$ and $\vec{v}\cdot\vec{H}$ commutes with every generator of the algebra and generates a one-dimensional abelian invariant subalgebra. But this is impossible because the whole algebra is semisimple.

Thus a compact semisimple finite dimensional Lie algebra has its structure determined by the dimensionality m of its Cartan subalgebra and the integers $p_{\alpha\beta}$ for each pair of the m simple roots, $\vec{\alpha}$ and $\vec{\beta}$. These determine the $\vec{\alpha}\cdot\vec{\beta}$, and thus the root vectors which exist. The procedure for determining which positive linear combinations of simple roots is described in Georgi page 107. Instead of repeating it, I will work out an example.

Let us classify all the simisimple Lie algebras of rank 2. There are only two simple roots, say $\vec{\alpha}$ and $\vec{\beta}$, and we may as well choose them such that $|\alpha|\leq|\beta|$.

We know that $-2\dfrac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}=p$, $-2\dfrac{\vec{\alpha}\cdot\vec{\beta}}{\beta^2}=p'$, where p and p' are nonnegative integers. If $\vec{\alpha}\cdot\vec{\beta}=0$, both $p=p'=0$, otherwise $\cos^2\theta = pp'/4$, so $pp'=1$, 2, or 3. As we took $|\alpha|\leq|\beta|$, $p\geq p'$, so the four posibilities are

$$\begin{array}{ll}
\text{(a)} & p=p'=0\\
\text{(b)} & p=p'=1\\
\text{(c)} & p=2,\ p'=1\\
\text{(d)} & p=3,\ p'=1
\end{array}$$

The first three posibilities are to be considered by you for homework. I will do case (d) now.

We have $-2\dfrac{\vec{\alpha}\cdot\vec{\beta}}{\alpha^2}=3$, $-2\dfrac{\vec{\alpha}\cdot\vec{\beta}}{\beta^2}=1$, so $\beta^2=3\alpha^2$ and $\cos\theta=-\sqrt{3}/2$,

$\theta = 150°$. To choose our root vectors, we note that the requirements
1) the killing form is a multiple of the identity
2) the subspace $\mathcal{H}$ is abelian, and left invariant by overall scale changes and rotations of the H_i's.
This means that we can change the scale and rotate the whole root diagram. The only proviso is that what had been positive may cease to remain so.

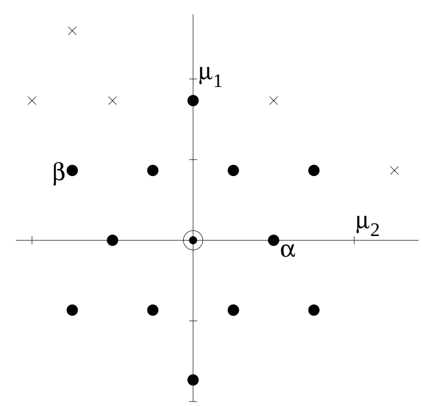
Choose $\vec{\alpha} = (0, 1)$ so $\vec{\beta} = \left(\sqrt{3}/2, -3/2\right)$. All roots must be of the form $\vec{\gamma} = m\vec{\alpha} + n\vec{\beta}$ with $m \geq 0$ and $n \geq 0$, or the negative of such a root. As $p = 3$, we are assured that $E_{\vec{\alpha}}$ can be applied three times to $E_{\vec{\beta}}$ before vanishing, giving

$$E_{\vec{\alpha}+\vec{\beta}}, E_{2\vec{\alpha}+\vec{\beta}}, E_{3\vec{\alpha}+\vec{\beta}},$$

but guaranteeing that $E_{4\vec{\alpha}+\vec{\beta}}$ does not exist. As $p' = 1$, $E_{\vec{\beta}}$ can be applied only once to $E_{\vec{\alpha}}$, so $E_{\vec{\alpha}+2\vec{\beta}}$ does not exist.

We still need to ask if $E_{\vec{\beta}}$ can be applied to $E_{2\vec{\alpha}+\vec{\beta}}$ or $E_{3\vec{\alpha}+\vec{\beta}}$ and give a new state. In both cases $E_{-\vec{\beta}}$ could not, because this would give a root parallel to $E_{\vec{\alpha}}$, so we know $q = 0$. Then as for any $\vec{\mu}$, $\dfrac{\vec{\mu}\cdot\vec{\beta}}{\beta^2} = \dfrac{q-p}{2}$, in this case the appropriate p is $-\frac{2}{3}\vec{\beta}\cdot(r\vec{\alpha}+\vec{\beta}) = r-2$, so $E_{2\vec{\alpha}+2\vec{\beta}}$ does not exist, but just one application of $E_{\vec{\beta}}$ to $E_{3\vec{\alpha}+\vec{\beta}}$ is allowed, giving $E_{3\vec{\alpha}+2\vec{\beta}}$, but not $E_{3\vec{\alpha}+3\vec{\beta}}$.

Can $E_{\vec{\alpha}}$ act on our new $E_{3\vec{\alpha}+2\vec{\beta}}$? $E_{-\vec{\alpha}}$ cannot, as $2\vec{\alpha}+2\vec{\beta}$ is not a root, so $q = 0$, and $\vec{\alpha}\cdot(3\vec{\alpha}+2\vec{\beta}) = 0$, so $p = 0$ and it can not. So we have found all the positive roots, and adding their negatives we have this pretty diagram with two Cartan basis vectors and 12 roots, giving the 14 dimensional algebra called G_2.



Roots diagram for G_2. The two generators of the Cartan subalgebra are indicated at the origin. The 12 roots are shown by dots, and the weights excluded as being roots in the argument are shown as crosses.