Chapter 11

S_k and Tensor Representations

(Ref: Schensted Part II)

If we have an arbitrary tensor with k indices $W^{i_1,\cdots,i_k}$ we can act on it with a permutation $P = \begin{pmatrix} 1 & 2 & \cdots & k \\ a & b & \cdots & \ell \end{pmatrix}$ so

$$(Pw)^{i_1,i_2,\cdots,i_k} = w^{i_a,i_b,\cdots,i_\ell}.$$

Consider the algebra $\mathcal{A}$ formed by taking arbitrary linear combinations of the different permutations, considered as operators acting on the space of k'th rank tensors. This algebra can be constructed for any group, particularly finite groups, and is called the **group algebra**. (this is **not** the Lie algebra!). Note that this sum of permutations makes sense only as operators on a vector space. It is not the composition of permutations. Also note that as $\mathcal{A}$ is an *algebra*[1], one can both add and multiply (by composition) elements in $\mathcal{A}$.

[1]**Definition:** An *algebra* consists of a vector space V over a field F, together with a binary operation of multiplication on the set V of vectors, such that for all $a \in F$ and $\alpha, \beta, \gamma \in V$, the following are satisfied:

1. $(a\alpha)\beta = a(\alpha\beta) = \alpha(a\beta)$
2. $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$
3. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

V is an *associative algebra over* F if, in addition,

4 $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in V$.

The group algebra is useful because it can extract the tensors of specified symmetry. First consider tensors of rank 2. Writing $\mathbb{1} = \frac{1}{2}(\mathbb{1} + (1\,2)) + \frac{1}{2}(\mathbb{1} - (1\,2))$ we can extract

$$\begin{aligned} s^{ij} &= \frac{1}{2}(\mathbb{1} + (1\,2))\, w^{ij} \\ a^{ij} &= \frac{1}{2}(\mathbb{1} - (1\,2))\, w^{ij} \end{aligned}$$

and $w^{ij} = s^{ij} + a^{ij}$ is a decomposition into a symmetric tensor and an anti-symmetric tensor.

The action of the permutations commutes with the SU(n) rotations on the tensors, so a constraint on a tensor of the form $Aw = 0$ for some $A \in \mathcal{A}$, if it holds for one state of an irreducible representation of SU(n), will hold on all states in that representation. Thus s and a are separate representations.

Now consider a rank 3 tensor w^{ijk}, and define

$$s^{ijk} = \frac{1}{6} \sum_{P \in S_3} P w^{ijk}$$

$$a^{ijk} = \frac{1}{6} \sum_{P \in S_3} (\operatorname{sign} P) P w^{ijk}$$

These are the totally symmetric and totally antisymmetric parts of w, but it is not all of w. For example, suppose $w^{112} = w^{121} = 1$, $w^{211} = -2$, all other components zero. Then s^{ijk} and a^{ijk} are both zero. The rest is related to the two-dimensional representation of S_3 (see homework #3, problem 1). In general, there will be operators in $\mathcal{A}$ associated with the different irreducible representations of S_k, which extract the corresponding irreducible representations of SU(n).

So we now turn to the problem of finding the irreducible representations of S_k.

11.1 Irreducible Representations of S_k

We know in general that the number of irreducible representations is the number of conjugacy classes. So let us begin with that.

Any element of S_k can be written as a product of disjoint cycles. For example, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = (1\,2\,3)(4\,5)$. This factorization is unique (remember $(1\,2\,3) = (2\,3\,1)$) up to the order of the factors, which commute because they are *disjoint* cycles.

Under conjugation by $P = \begin{pmatrix} 1 & 2 & \cdots & k \\ P_1 & P_2 & \cdots & P_k \end{pmatrix}$ a cycle simply has its elements permuted. Thus $P(i\,j\,k)P^{-1} = (P_i\,P_j\,P_k)$. This is true for products of cycles as well. Thus two permutations whose descriptions in terms of disjoint cycles contain the same number of cycles of each length are conjugate, and only those are. We describe the conjugacy class of elements describable in terms of disjoint cycles, I_k of length ℓ_k, as $\left(\ell_1^{i_1}\,\ell_2^{i_2}\,\cdots\right)$. Including one-cycles for any element left unmoved, we have $\sum_m i_m \ell_m = k$.
Example: S_3

permutations		class	
$1\!\!1$	$\in$	(1^3)	
$(1\,2) = (1\,2)(3)$	$\in$	$(2,1)$	
$(2\,3)$, $(1\,3)$	$\in$	$(2,1)$	as well
$(1\,2\,3)$, $(1\,3\,2)$	$\in$	(3)	

There is one conjugacy class for each partition of k. A partition of an integer k is an unordered set of positive integers, possibly with repeats, which add to k.
Example: How many classes[2] are there in S_5?

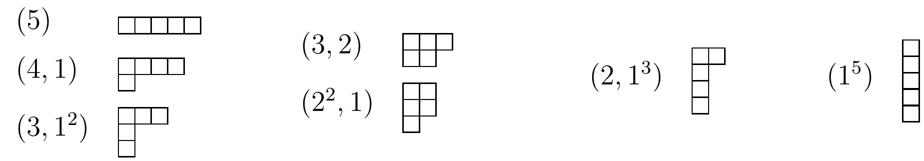
$$(5);\quad (4,1);\quad (3,1,1)=(3,1^2);\quad (3,2);\quad (2^2,1);\quad (2,1^3);\quad (1^5)$$

answer: 7.

Thus we also know that there are that many irreducible representations, although there is not a straightforward correspondance between the representations and the conjugacy classes.

Define a **Young graph** for S_k as a set of k boxes arranged, left-justified, in rows each of which is no longer than the preceeding. The lengths of the rows provide a partition of k. So

[2]The number of partitions of n is given by the partition function of number theory, $p(n)$. There are other things called partition functions, especially Z of statistical mechanics, which is different. The number-theory one, also called the integer partition function, arises also in counting states in string theory. It has the fascinating property that $p(k)$ has the generating function $\prod_{k=1}^\infty (1-x^k)^{-1} = \sum_{k=0} p(k)x^k$, where we say $p(0)=1$.



There is one irreducible representation of S_k corresponding to each Young graph.

A **Young tableau** is a Young graph with the numbers $1, 2, \cdots, k$ inserted in the boxes in some order, for example $\tau = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$.
For each tableau we define an element of the group algebra, $P_\tau = \sum P$, where the sum is over those permutations which permute the numbers within each row but do not move them from one row to another. Here

$$P_\tau = [1\!\!1 + (2\,3) + (3\,4) + (2\,4) + (2\,3\,4) + (2\,4\,3)]\,[1\!\!1 + (5\,1)]\,,$$

which includes 12 of the 120 permutations in S_5.
We also associate $Q_\tau = \sum(\text{sign}\,P)P$ where the sum includes only permutations which permute numbers in the same column but don't move numbers from one column to another. Thus

$$Q_\tau = [1\!\!1 - (2\,5)]\,[1\!\!1 - (1\,3)]\,.$$

Finally we define the **Young operator** $Y_\tau = Q_\tau P_\tau$.

We see that the way to get a totally symmetric rank 5 tensor is to apply $Y_{\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline\end{array}}$ to an arbitrary one while you get a totally antisymmetric tensor by applying $Y_{\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}}$, with the numbers in any order in the boxes.

The Y_τ corresponding to any Young tableau τ is almost, but not quite, the element of the group algebra we want to extract irreducible representations. We find a related set of basis vectors in the group algebra by using the representations of S_k. Define

$$e^\eta_{ij} = \frac{\ell_\eta}{k!} \sum_{P\in S_k} \Gamma^\eta_{ji}(P^{-1})P,$$

where η is the Young graph corresponding to an irreducible representation of S_k, and ℓ_η is the dimension of that representation. The sum is over all the permutations.

For $\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}$, $\ell_\eta = 1$, $\Gamma = 1$, and $e_{11}^{\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}} = Y_{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}}$ which is also equal to any other Young operator for a tableau in $\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}$.

For $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$, $\ell_\eta = 1$, $\Gamma = \operatorname{sign} P$, $e_{11}^{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}} = Y_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}$. Other Young operators in $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$ differ only in sign, *e.g.* $Y_{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array}} = \operatorname{sign}(1\,2) \cdot Y_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}} = -Y_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}}$.

The e's have some marvelous properties. They form vector spaces transforming as irreducible representations under S_k separately from the right and from the left: For[3] $Q \in S_k$,

$$\begin{aligned} Qe^\eta_{ij} &= \frac{\ell_\eta}{k!}\sum_P \Gamma^\eta_{ji}\left(P^{-1}\right)QP = \frac{\ell_\eta}{k!}\sum_m \Gamma^\eta_{mi}(Q)\sum_P \Gamma^\eta_{jm}\left(P^{-1}Q^{-1}\right)QP \\ &= \frac{\ell_\eta}{k!}\sum_m \Gamma^\eta_{mi}(Q)\sum_R \Gamma^\eta_{jm}\left(R^{-1}\right)R = \sum_m \Gamma^\eta_{mi}(Q)e^\eta_{mj} \end{aligned}$$

where we again used the rearrangement theorem. Thus Q acts just the way you'ld expect for a basis vector e_i of representation η to transform, for each fixed j.

From the other side, $e^\eta_{ij}Q = \sum_m \Gamma^\eta_{jm}(Q)e^\eta_{im}$.

We say that the set e^η_{ij} is a two sided ideal (or invariant subalgebra) of the group algebra over S_k.

This gives the e's an interesting algebra:

$$\begin{aligned} e^\eta_{ij}e^{\eta'}_{mn} &= \frac{\ell_\eta\ell_{\eta'}}{(k!)^2}\sum_{P,P'\in S_k}\Gamma^\eta_{ji}\left(P^{-1}\right)\Gamma^{\eta'}_{nm}\left(P'^{-1}\right)PP' \\ &= \frac{\ell_\eta\ell_{\eta'}}{(k!)^2}\sum_{P,R\in S_k}\Gamma^\eta_{ji}\left(P^{-1}\right)\sum_p \Gamma^{\eta'}_{np}\left(R^{-1}\right)\Gamma^{\eta'}_{pm}\left(P\right)R \\ &= \frac{\ell_\eta}{k!}\sum_{P\in S_k}\Gamma^{\eta *}_{ij}(P)\sum_p \Gamma^{\eta'}_{pm}\left(P\right)e^{\eta'}_{pn} \qquad \text{by unitarity} \\ &= \delta_{\eta\eta'}\delta_{jm}e^\eta_{in} \qquad \text{by the great orthogonality theorem} \end{aligned}$$

We may also show that the diagonal elements e^η_{ii} form a decomposition of the identity. From the great orthogonality theorem "transposed" ,

$$\delta_{GG'} = \sum_{ij\eta}\frac{\ell_\eta}{k!}\Gamma^{\eta *}_{ij}(G')\Gamma^\eta_{ij}(G)$$

[3]Note P and Q are any elements of S_k, and are not related to P_τ and Q_τ defined earlier.

we can write the identity element of S_k as

$$\begin{aligned} \mathbb{1} &= \sum_G \delta_{G^{-1},\mathbf{1}}G = \sum_{ij\eta G}\frac{\ell_\eta}{k!}\Gamma^{\eta *}_{ij}(\mathbb{1})\Gamma^\eta_{ij}(G^{-1})G \\ &= \sum_{i\eta G}\frac{\ell_\eta}{k!}\Gamma^\eta_{ii}(G^{-1})G = \sum_{\eta i} e^\eta_{ii} \end{aligned}$$

so

$$\mathbb{1} = \sum_{\eta i} e^\eta_{ii}$$

Thus the whole algebra is spanned by these two sided ideals. In particular, the Y_τ are contained in the corresponding e^η_{ij} (an ℓ^2_η dimensional algebra).

In fact, the space spanned by e^η_{ij} is also spanned by $Q_i s_{ij} P_j$, where Q_i and P_i are the antisymmetrizers and symmetrizers of a set of **standard tableaux** for η, which means tableaux in which the numbers increase left to right in each row, and also top to bottom in each column. Thus $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ are standard tableaux, but $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ are not. Here s_{ij} is the permutation such that $\tau_i = s_{ij}\tau_j$. Each of these spaces has dimension ℓ^2_η, with ℓ_η equal to the number of standard tableaux, so

The dimension of Γ^η is the number of standard tableaux of η.

Counting all possibilities is tedious, so we have a magic formula in terms of **hooks**.

For each box b in a Young graph with k boxes, define the hook of b, $g_b = 1$ plus the number of boxes directly to the right plus the number of boxes directly beneath. Then

$$\ell_\eta = \frac{k!}{\prod_b g_b}.$$

Example: In the Young graph I have placed the corresponding hooks (this is not a Young tableau)

$$\begin{array}{|c|c|c|c|}\hline 7 & 5 & 3 & 2 \\ \hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad \ell = \frac{11!}{7\cdot 6\cdot 5\cdot 4\cdot 3^2\cdot 2^2\cdot 1^3} = 1320.$$

It would be hard to count this explicitly. For our more reasonable case $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$ gives $\ell = \frac{3!}{3} = 2$.

11.2 Representations of SU(n)

We now turn to the extraction of arbitary representations of SU(n). Georgi discusses the fundamental weights of SU(n), and shows that an arbitrary representation can be found from a tensor product of an adequate number of defining representations. The problem is to extract from the tensor product of k defining representations $\otimes\mathbf{N}^k$ the irreducible pieces. We have seen that this can be done by demanding that elements of the permutation group algebra vanish. If we impose $e^\eta_{ii}w = 0$ for all η and i save one, that is equivalent to projecting out our representation

$$T^{a_1 a_2 \cdots a_k} = (e^\eta_{ii} w)^{a_1 a_2 \cdots a_k} \qquad \text{no sum on } i$$

for one particular representation η and one basis vector i.

The different i generate equivalent representations. The different η's, however, each correspond to a different (inequivalent) representation of SU(n).

Before doing more formal arguments, we will do an example. Consider three spin $\frac{1}{2}$ objects, or the tensor product of three defining representations of SU(2). We will extract from this 8 dimensional state space the piece $e^{\yng(2,1)}_{11}$. From the problem you did for homework (#3, problem 1),

$$\begin{aligned} e^{\yng(2,1)}_{11} &= \frac{2}{6}\sum_P \Gamma^{e^{\yng(2,1)}_{11}}\left(P^{-1}\right) P \\ &= \frac{1}{3}\left(\mathbb{1} + (1\,2) - \frac{1}{2}(2\,3) - \frac{1}{2}(1\,3) - \frac{1}{2}(1\,2\,3) - \frac{1}{2}(1\,3\,2)\right), \end{aligned}$$

Let this act on the basis vectors which we expand as $\uparrow = e_1$, $\downarrow = e_2$.

v	$e^{\yng(2,1)}_{11} v$		
$\uparrow\uparrow\uparrow$	$\frac13\uparrow\uparrow\uparrow+\frac13\uparrow\uparrow\uparrow-\frac16\uparrow\uparrow\uparrow-\frac16\uparrow\uparrow\uparrow-\frac16\uparrow\uparrow\uparrow-\frac16\uparrow\uparrow\uparrow$	$=$	0
$\uparrow\uparrow\downarrow$	$\frac13\uparrow\uparrow\downarrow+\frac13\uparrow\uparrow\downarrow-\frac16\uparrow\downarrow\uparrow-\frac16\downarrow\uparrow\uparrow-\frac16\downarrow\uparrow\uparrow-\frac16\uparrow\downarrow\uparrow$	$=$	$\frac23\uparrow\uparrow\downarrow-\frac13\uparrow\downarrow\uparrow-\frac13\downarrow\uparrow\uparrow$
$\uparrow\downarrow\uparrow$	$\frac13\uparrow\downarrow\uparrow+\frac13\downarrow\uparrow\uparrow-\frac16\uparrow\uparrow\downarrow-\frac16\uparrow\downarrow\uparrow-\frac16\uparrow\uparrow\downarrow-\frac16\downarrow\uparrow\uparrow$	$=$	$-\frac13\uparrow\uparrow\downarrow+\frac16\uparrow\downarrow\uparrow+\frac16\downarrow\uparrow\uparrow$
$\downarrow\uparrow\uparrow$	$\frac13\downarrow\uparrow\uparrow+\frac13\uparrow\downarrow\uparrow-\frac16\downarrow\uparrow\uparrow-\frac16\uparrow\uparrow\downarrow-\frac16\uparrow\downarrow\uparrow-\frac16\uparrow\uparrow\downarrow$	$=$	$-\frac13\uparrow\uparrow\downarrow+\frac16\uparrow\downarrow\uparrow+\frac16\downarrow\uparrow\uparrow$
$\uparrow\downarrow\downarrow$	$\frac13\uparrow\downarrow\downarrow+\frac13\downarrow\uparrow\downarrow-\frac16\uparrow\downarrow\downarrow-\frac16\downarrow\downarrow\uparrow-\frac16\downarrow\uparrow\downarrow-\frac16\downarrow\downarrow\uparrow$	$=$	$\frac16\uparrow\downarrow\downarrow+\frac16\downarrow\uparrow\downarrow-\frac13\downarrow\downarrow\uparrow$
$\downarrow\uparrow\downarrow$	$\frac13\downarrow\uparrow\downarrow+\frac13\uparrow\downarrow\downarrow-\frac16\downarrow\downarrow\uparrow-\frac16\downarrow\uparrow\downarrow-\frac16\downarrow\downarrow\uparrow-\frac16\uparrow\downarrow\downarrow$	$=$	$\frac16\uparrow\downarrow\downarrow+\frac16\downarrow\uparrow\downarrow-\frac13\downarrow\downarrow\uparrow$
$\downarrow\downarrow\uparrow$	$\frac13\downarrow\downarrow\uparrow+\frac13\downarrow\downarrow\uparrow-\frac16\downarrow\uparrow\downarrow-\frac16\uparrow\downarrow\downarrow-\frac16\uparrow\downarrow\downarrow-\frac16\downarrow\uparrow\downarrow$	$=$	$-\frac13\uparrow\downarrow\downarrow-\frac13\downarrow\uparrow\downarrow+\frac23\downarrow\downarrow\uparrow$
$\downarrow\downarrow\downarrow$	$\frac13\downarrow\downarrow\downarrow+\frac13\downarrow\downarrow\downarrow-\frac16\downarrow\downarrow\downarrow-\frac16\downarrow\downarrow\downarrow-\frac16\downarrow\downarrow\downarrow-\frac16\downarrow\downarrow\downarrow$	$=$	0

Notice this only results in one state of $J_z = \frac{1}{2}$ and one of $J_z = -\frac{1}{2}$. So $e^{\yng(2,1)}_{11}$ projects out a 2-dimensional $s = \frac{1}{2}$ state. $e^{\yng(2,1)}_{22}$ would project out an orthogonal spin $\frac{1}{2}$. Thus the tensor product of three spin 1/2's is a spin 3/2 (the totally symmetric part, $e^{\yng(3)}_{11}$) and two spin 1/2 representations, $2 \times 2 \times 2 = 4 + 2 + 2$.

Having completed this trivial but tedious example of the simple case of SU(2) and $\yng(2,1)$, we are ready for some abstract reasoning.

Now we consider the general case of $\mathbf{N}^k$. The basis vectors which are mixed by the permutations are only those with the same number of indices equal to 1, and the same number equal to 2, *etc.*. Consider the subspace with r_i of the indices equal to i, with $\sum r_i = k$, each $r_i = 1, \ldots, N$.

This subspace $\mathcal{S}^{\vec r}$ is spanned by the basis vector

$$e = \underbrace{e_1 \otimes e_1 \cdots \otimes e_1}_{r_1 \text{ times}} \otimes \underbrace{e_2 \cdots \otimes e_2}_{r_2 \text{ times}} \cdots \otimes \underbrace{e_N \cdots \otimes e_N}_{r_N \text{ times}},$$

together with all permutations $P\,e$, for $P \in S_k$. If all the indices are different, all $r_i = 0$ or 1, all of the permutations are inequivalent, and we get a $k!$ dimensional space. But if the r_i's are not all ≤ 1, there is a subgroup $\mathcal{P} \subset S_k$ with $B\,e = e$ for $B \in \mathcal{P}$. In fact, $\mathcal{P} = S_{r_1} \times S_{r_2} \times \cdots S_{r_N}$.

Let $P_{\mathcal{P}} = \sum_{B\in\mathcal{P}} B$ which is a element of the group algebra $\mathcal{A}$. Then while the subspace $\mathcal{S}^{\vec r}$ is spanned by $\{P\,e | P \in S_k\}$ it is also spanned by $\{P P_{\mathcal{P}}\,e | P \in S_k\}$.

We now want to extract from $\mathcal{S}^{\vec r}$ the piece projected out by e^η_{ii}. The products $\{e^\eta_{ii}P\}$ for all P are just sums of multiples of e^η_{ij}, for all j (by p. 125) so we want to know the dimension of the space $\{e^\eta_{ij}P_\mathcal{P}|j=1,\ell_\eta\}$. As e^η is a two-sided ideal, this space is $\langle\sum_k e^\eta_{ik}b_k\rangle$, so the dimensionality depends on the constraints on b_k. If they were all independent, they would form an ℓ_η dimensional space. But there are constraints. For $B\in\mathcal{P}$, $P_\mathcal{P}B=P_\mathcal{P}$. Let's be more explicit:

$$e^\eta_{ij}P_\mathcal{P}=\sum_n e^\eta_{in}b_{nj}=e^\eta_{ij}P_\mathcal{P}B=\sum_n b_{nj}e^\eta_{in}B=\sum_{nm}\Gamma^\eta_{nm}(B)e^\eta_{im}b_{nj}.$$

The e^η_{im} are linearly independent, so $b_{nj}=\sum_m b_{mj}\Gamma^\eta_{mn}(B)$, for $B\in\mathcal{P}$. To find out how many degrees of freedom survive this constraint for each j, observe that $\Gamma^\eta_{mn}(B)$ forms a reducible representation of the subgroup $\mathcal{P}$. So we can write

$$\Gamma^\eta(B)=U\bigoplus_\epsilon\Gamma^\epsilon(B)\,U^{-1}$$

where Γ^ϵ are irreducible representations of $\mathcal{P}$. Now if $c=bU$,

$$b=b\Gamma^\eta(B)=bU\bigoplus\Gamma^\epsilon(B)U^{-1}\Longrightarrow c=c\left(\bigoplus\Gamma^\epsilon(B)\right).$$

The vector c breaks up into pieces for each representation ϵ, with $c^\epsilon\Gamma^\epsilon(B)=c^\epsilon$ for all $B\in\mathcal{P}$. This is possible for nonzero c only if ϵ is the identity representation, as the representations are irreducible.

Therefore the dimensionality of the space $e^\eta_{ii}Pe$ is the number of times, γ_η, that the identity representation of $\mathcal{P}$ is contained in Γ^η.

But the number of times the representation i is contained in Γ^η is

$$\gamma_\eta=a_{i=\mathbf{1}}=\frac{1}{g_\mathcal{P}}\sum_{B\in\mathcal{P}}\chi^{i\,*}(B)\chi^\eta(B)=\frac{1}{g_\mathcal{P}}\sum_{B\in\mathcal{P}}\chi^\eta(B)$$

for i=identity, where $g_\mathcal{P}$ is the number of elements in $\mathcal{P}$, which is $\prod r_i!$

Example: $\eta=\yng(2,1)$. From homework, and recalling $\chi=\operatorname{Tr}\Gamma$,

$$\begin{aligned}\chi&=2 &&\text{for}\quad B=1\!\!1\\ \chi&=0 &&\text{for}\quad B=(1\,2),(1\,3),\ \text{or}\ (2\,3)\\ \chi&=-1 &&\text{for}\quad B=(1\,2\,3),\ \text{or}\ (1\,3\,2)\end{aligned}$$

Consider the space starting from $e_1\otimes e_1\otimes e_2$.

$$\mathcal{P}=\{1\!\!1,(1\,2)\},\qquad g_\mathcal{P}=2,\qquad \gamma_\eta=\frac12(2+0)=1$$

so e_{11} generates only one state from the three-dimensional space $\mathcal{S}^{\vec r}$.
From $e_1\otimes e_1\otimes e_1$, $\mathcal{P}=S_3$, $g_\mathcal{P}=6$, $\gamma_\eta=\frac16(2-1-1)=0$ so we get no state here.
If all vectors are unequal, say $e_1\otimes e_2\otimes e_3$ for SU($n>2$), $\mathcal{P}=1\!\!1$, $g_\mathcal{P}=1$, $\gamma_\eta=\frac21=2$.

For SU(N), there are N states of the form $e_i\otimes e_i\otimes e_i$, not contributing anything to $\eta=\yng(2,1)$. There are $N(N-1)$ states $e_i\otimes e_i\otimes e_j$ with $i\neq j$, each contributing one state, so from these we get $N(N-1)$ states. There are also $N(N-1)(N-2)/6$ states of the form $e_i\otimes e_j\otimes e_k$, with $i<j<k$, each contributing 2 states, so the dimension of $\yng(2,1)$ is

$$N(N-1)+\frac13N(N-1)(N-2)=\frac{N(N^2-1)}{3}=\begin{cases}2\text{ for }N=2\\8\text{ for }N=3\end{cases}$$

Let's work another example, $\yng(3,1)$ for SU(N). As we need the characters for this representation, let's take them from Schensted: $\chi=3$ for $1\!\!1\in(1^4)$, [1 element]; $\chi=0$ for $(3,1)$, [8 elements]; $\chi=1$ for $(2,1^2)$, [6 elements]; $\chi=-1$ for $(2,2)$, [3 elements] and for (4), [6 elements].

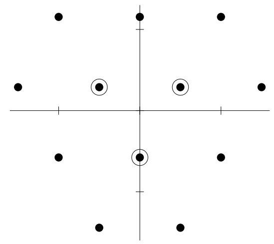
Enumerating the basis states in the various partitions, and multiplying $\chi(B)$ by their number for those within $\mathcal{P}$, we find

indices	subspace	$\mathcal{P}$	γ_η $\quad 1^4\quad(3,1)\quad(2,1^2)\quad(2,2)\quad(4)$
all i	$e_ie_ie_ie_i$	S_4	$\frac{1}{24}\Big(1\cdot3+8\cdot0+6\cdot1+3\cdot(-1)+6\cdot(-1)\Big)=0$
$i\neq j$	$e_ie_ie_ie_j$	S_3	$\frac16\Big(1\cdot3+2\cdot0+3\cdot1+0\cdot(-1)+0\cdot(-1)\Big)=1$
$i<j$	$e_ie_ie_je_j$	$S_2{\times}S_2$	$\frac14\Big(1\cdot3+0\cdot0+2\cdot1+1\cdot(-1)+0\cdot(-1)\Big)=1$
$i{\neq}j{<}k{\neq}i$	$e_ie_ie_je_k$	S_2	$\frac12\Big(1\cdot3+0\cdot0+1\cdot1+0\cdot(-1)+0\cdot(-1)\Big)=2$
all $\neq$	$e_ie_je_ke_\ell$	$(1\!\!1)$	$\frac11\Big(1\cdot3+0\cdot0+0\cdot1+0\cdot(-1)+0\cdot(-1)\Big)=3$

Rewriting this with only the results for γ_η to allow room for counting index choices and states, we have

indices	subspace	$\mathcal{P}$	γ_η	# index choices	# states
all i	$e_ie_ie_ie_i$	S_4	0	N	0
$i \neq j$	$e_ie_ie_ie_j$	S_3	1	$N(N-1)$	$N(N-1)$
$i < j$	$e_ie_ie_je_j$	$S_2 \times S_2$	1	$N(N-1)/2$	$N(N-1)/2$
$i \neq j < k \neq i$	$e_ie_ie_je_k$	S_2	2	$N(N-1)(N-2)/2$	$N(N-1)(N-2)$
all $\neq$	$e_ie_je_ke_\ell$	$(\mathbb{1})$	3	$\binom{N}{4}$	$3\binom{N}{4}$

So the total dimensionality of $\ydiagram{3,1}$ for SU(N) is



$$0 + N(N-1) + \frac{N(N-1)}{2} + N(N-1)(N-2) + 3\,\frac{N(N-1)(N-2)(N-3)}{4!}$$

$$= N(N-1)\left(\frac{3}{2} + (N-2) + \frac{1}{8}(N-2)(N-3)\right) = \frac{(N+2)!}{8\,(N-2)!}.$$

For SU(3), $N = 3$, the total dimension is

$$5!/(8 \cdot 1!) = 15.$$

We now know how to extract the irreducible representations or just to count their dimensionality. Now it is time for magic.

The number γ_η of states extracted by e_{ii}^η from $\mathcal{S}^{\vec{r}}$, the space spanned by $P\left(\bigotimes e_i^{r_i}\right)$ by all $P \in S_k$, is given by the number of ways one can place r_1 1's, r_2 2's, $\cdots$ in the Young graph so that in each row the numbers do not decrease, and in each column they increase. This is called a **permissible placement**.

To see how this works, let's check it out on $\ydiagram{3,1}$ for SU(N). For $r_1 = 4$ there is no way to avoid two 1's in the same column, so $\gamma = 0$.
For $r_1 = 3$ and $r_2 = 1$, the 2 has to be in the second row, so there is only one way, $\gamma = 1$.
For $r_1 = r_2 = 2$, the only possibility is $\ytableaushort{112,2}$, $\gamma = 1$.
For $r_1 = 2$, $r_2 = r_3 = 1$, we have $\ytableaushort{112,3}$ and $\ytableaushort{113,2}$, so $\gamma = 2$.
For $r_1 = r_2 = r_3 = r_4 = 1$, we have $\ytableaushort{123,4}$, $\ytableaushort{134,2}$, $\ytableaushort{124,3}$, and $\gamma = 3$.

Note in our previous method, it was clear that these numbers only depended on the set $\{r_i\}$ and not on the order. This is now not obvious.

Consider $r_1 = 1, r_2 = 2, r_4 = 1$. Then we have $\ytableaushort{124,2}$ and $\ytableaushort{122,4}$, so again $\gamma = 2$, as for $r_1 = 2$, $r_2 = r_3 = 1$.

Now to count the dimensionality of an irreducible representation of SU(N) belonging to the Young graph η, we must sum, over all choices r_i, the corresponding γ_η. But for each choice of r_i the γ_η is the number of ways of placing the indices in the graph in a permissible fashion. So the dimension of the full irreducible representation of SU(N) is the number of ways of placing k integers, chosen from $1, 2, , \ldots N$ (repeats allowed) in a permissible fashion in η.

Example $\ydiagram{2,1}$ for SU(3): $\ytableaushort{11,2}$, $\ytableaushort{11,3}$, $\ytableaushort{12,2}$, $\ytableaushort{12,3}$, $\ytableaushort{13,2}$, $\ytableaushort{13,3}$, $\ytableaushort{22,3}$, $\ytableaushort{23,3}$, for a total of 8 dimensions.

There is an easier method of finding the dimensionality. For each box, associate the value (N + column number − row number). Then divide by the hook of that box. The dimension of the representation is the product of these quotients over all the boxes.

$$\text{Examples:}\quad \text{Dim}\ \ydiagram{2,1} = \begin{ytableau} N & N{+}1 \\ N{-}1 \end{ytableau} \Big/ \begin{ytableau} 3 & 1 \\ 1 \end{ytableau} = \frac{N\left(N^2-1\right)}{3}$$

$$\text{Dim}\ \ydiagram{3,1} = \begin{ytableau} N & N{+}1 & N{+}2 \\ N{-}1 \end{ytableau} \Big/ \begin{ytableau} 4 & 2 & 1 \\ 1 \end{ytableau} = \frac{(N+2)!}{8(N-2)!}$$

Note: If the first column of a graph has N boxes, the hook of each box in column 1 is equal to the (N + column number − row number) of the last box in that row. Thus eliminating the first row does not change the dimension. In fact, it does not change the representation either. This is because a totally antisymmetric tensor with N indices is invariant.

Thus in SU(2), $\ydiagram{2,1} = \ydiagram{1}$, as we saw in detail. It also means that for SU(N), we needn't consider representations with N or more rows (except perhaps to indicate the identity representation by one column of N boxes).