Chapter 11

\( S_k \) and Tensor Representations

(Ref: Schensted Part II)

If we have an arbitrary tensor with \( k \) indices \( W_{i_1, \ldots, i_k} \), we can act on it with a permutation \( P = \begin{pmatrix} 1 & 2 & \cdots & k \\ a & b & \cdots & \ell \end{pmatrix} \) so

\[
(PW)_{i_1, i_2, \ldots, i_k} = w_{ia, ib, \ldots, \ell}.
\]

Consider the algebra \( \mathcal{A} \) formed by taking arbitrary linear combinations of the different permutations, considered as operators acting on the space of \( k \)'th rank tensors. This algebra can be constructed for any group, particularly finite groups, and is called the group algebra. (This is not the Lie algebra!)

Note that this sum of permutations makes sense only as operators on a vector space. It is not the composition of permutations. Also note that as \( \mathcal{A} \) is an algebra\(^1\), one can both add and multiply (by composition) elements in \( \mathcal{A} \).

\(^1\)Definition: An algebra consists of a vector space \( V \) over a field \( F \), together with a binary operation of multiplication on the set \( V \) of vectors, such that for all \( a \in F \) and \( \alpha, \beta, \gamma \in V \), the following are satisfied:

1. \( (aa)\beta = a(a\beta) = a(a\beta) \)
2. \( (a + \beta)\gamma = a\gamma + \beta\gamma \)
3. \( a(\beta + \gamma) = a\beta + a\gamma \)

\( V \) is an associative algebra over \( F \) if, in addition,

4. \( (a\beta)\gamma = a(\beta\gamma) \) for all \( \alpha, \beta, \gamma \in V \).

The group algebra is useful because it can extract the tensors of specified symmetry. First consider tensors of rank 2. Writing \( \mathbb{I} = \frac{1}{2} (\mathbb{I} + (1 2)) + \frac{1}{2} (\mathbb{I} - (1 2)) \) we can extract

\[
s_{ij} = \frac{1}{2} (\mathbb{I} + (1 2)) w^{ij}
\]

and

\[
a_{ij} = \frac{1}{2} (\mathbb{I} - (1 2)) w^{ij}
\]

and \( s_{ij} + a_{ij} \) is a decomposition into a symmetric tensor and an antisymmetric tensor.

The action of the permutations commutes with the SU(\( n \)) rotations on the tensors, so a constraint on a tensor of the form \( Aw = 0 \) for some \( A \in \mathcal{A} \), if it holds for one state of an irreducible representation of SU(\( n \)), will hold on all states in that representation. Thus \( s \) and \( a \) are separate representations.

Now consider a rank 3 tensor \( w^{ijk} \), and define

\[
s_{ijk} = \frac{1}{6} \sum_{P \in S_3} P w^{ijk}
\]

\[
a_{ijk} = \frac{1}{6} \sum_{P \in S_3} (\text{sign } P) P w^{ijk}
\]

These are the totally symmetric and totally antisymmetric parts of \( w \), but it is not all of \( w \). For example, suppose \( w^{112} = w^{121} = 1, w^{211} = -2, \) all other components zero. Then \( s_{ijk} \) and \( a_{ijk} \) are both zero. The rest is related to the two-dimensional representation of \( S_3 \) (see homework #3, problem 1). In general, there will be operators in \( \mathcal{A} \) associated with the different irreducible representations of \( S_k \), which extract the corresponding irreducible representations of SU(\( n \)).

So we now turn to the problem of finding the irreducible representations of \( S_k \).

11.1 Irreducible Representations of \( S_k \)

We know in general that the number of irreducible representations is the number of conjugacy classes. So let us begin with that.

Any element of \( S_k \) can be written as a product of disjoint cycles. For example, \( \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = (1 2 3)(4 5) \). This factorization is unique (re-
member \((123)=(231)\) up to the order of the factors, which commute because they are disjoint cycles.

Under conjugation by \(P = \begin{pmatrix} 1 & 2 & \cdots & k \\
1 \ P_1 & 2 \ P_2 & \cdots & k \ P_k \end{pmatrix}\) a cycle simply has its elements permuted. Thus \(P(i \ j \ k)P^{-1} = (P_i \ P_j \ P_k)\). This is true for products of cycles as well. Thus two permutations whose descriptions in terms of disjoint cycles contain the same number of cycles of each length are conjugate, and only those are. We describe the conjugacy class of elements describable in terms of disjoint cycles, \(I_k\) of length \(\ell_k\), as \((\ell_1 \ell_2 \cdots)\). Including one-cycles for any element left unmoved, we have \(\sum_m \ell_m = k\).

Example: \(S_3\)

<table>
<thead>
<tr>
<th>permutations</th>
<th>class</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>((1^3))</td>
</tr>
<tr>
<td>((12))</td>
<td>((2,1))</td>
</tr>
<tr>
<td>((23),(13))</td>
<td>((2,1))  as well</td>
</tr>
<tr>
<td>((123),(132))</td>
<td>((3))</td>
</tr>
</tbody>
</table>

There is one conjugacy class for each partition of \(k\). A partition of an integer \(k\) is an unordered set of positive integers, possibly with repeats, which add to \(k\).

Example: How many classes\(^2\) are there in \(S_5\)?

\((5)\); \((4,1)\); \((3,1,1) = (3,1^2)\); \((3,2)\); \((2^2,1)\); \((2,1^3)\); \((1^5)\)

answer: 7.

Thus we also know that there are that many irreducible representations, although there is not a straightforward correspondance between the representations and the conjugacy classes.

Define a Young graph for \(S_k\) as a set of \(k\) boxes arranged, left-justified, in rows each of which is no longer than the preceding. The lengths of the rows provide a partition of \(k\). So

\(^2\)The number of partitions of \(n\) is given by the partition function of number theory, \(p(n)\). There are other things called partition functions, especially \(Z\) of statistical mechanics, which is different. The number-theory one, also called the integer partition function, arises also in counting states in string theory. It has the fascinating property that \(p(k)\) has the generating function \(\prod_{k=1}^\infty (1-x^k)^{-1} = \sum_{k=0} p(k) x^k\), where we say \(p(0) = 1\).
For $\ell_\eta = 1, \Gamma = 1$, and $c_{i_1} = \boxed{1\, 2\, 3\, 4\, 5}$, which is also equal to any other Young operator for a tableau in $\boxed{1\, 2\, 3\, 4\, 5}$.

For $\ell_\eta = 1, \Gamma = \text{sign } P$, $c_{i_1} = \boxed{1\, 2\, 3\, 4\, 5}$. Other Young operators in $\boxed{1\, 2\, 3\, 4\, 5}$ differ only in sign, e.g. $Y_1 = \text{sign}(1\, 2) \cdot Y_1 = -Y_1$.

The $e$'s have some marvelous properties. They form vector spaces transforming as irreducible representations under $S_k$ separately from the right and from the left: For $3 \in S_k$,

$$Q e_{ij}^\eta = \frac{\ell_\eta}{k!} \sum_P \Gamma^\eta_{ji} (P^{-1}) Q P = \frac{\ell_\eta}{k!} \sum_{m} \Gamma^\eta_{mi}(Q) \sum_P \Gamma^\eta_{jm} (P^{-1} Q^{-1}) Q P$$

$$= \frac{\ell_\eta}{k!} \sum_{m} \Gamma^\eta_{mi}(Q) \sum_R \Gamma^\eta_{jm} (R^{-1}) R = \sum_{m} \Gamma^\eta_{mi}(Q) e^\eta_{mj}$$

where we again used the rearrangement theorem. Thus $Q$ acts just the way you'd expect for a basis vector $e_i$ of representation $\eta$ to transform, for each fixed $j$.

From the other side, $e_{ij}^\eta Q = \sum_{m} \Gamma^\eta_{jm}(Q) e^\eta_{im}$.

We say that the set $e_{ij}^\eta$ is a two sided ideal (or invariant subalgebra) of the group algebra over $S_k$.

This gives the $e$'s an interesting algebra:

$$e_{ij}^\eta e_{mn}^\ell = \frac{\ell_\eta \ell_\ell}{(k!)^2} \sum_{P,P' \in S_k} \Gamma^\eta_{ji} (P^{-1}) \Gamma^{\ell}_{mn} (P'^{-1}) PP'$$

$$= \frac{\ell_\eta \ell_\ell}{(k!)^2} \sum_{P,R \in S_k} \Gamma^\eta_{ji} (P^{-1}) \Gamma^{\ell}_{np} (R^{-1}) \Gamma^{\ell}_{pm} (P) R$$

$$= \frac{\ell_\eta}{k!} \sum_{P \in S_k} \Gamma^{\ell}_{ji} (P) \Gamma^{\ell}_{pm} (P) e^\ell_{pn} \quad \text{by unitarity}$$

$$= \delta_{\eta \ell} \delta_{jm} e^\eta_{in} \quad \text{by the great orthogonality theorem}$$

We may also show that the diagonal elements $e_{ii}^\eta$ form a decomposition of the identity. From the great orthogonality theorem “transposed”,

$$\delta_{GG'} = \sum_{ij\eta} \frac{\ell_\eta}{k!} \Gamma^\eta_{ij} (G^*) \Gamma^\eta_{ij} (G)$$

4Note $P$ and $Q$ are any elements of $S_k$, and are not related to $P_\tau$ and $Q_\tau$ defined earlier.

we can write the identity element of $S_k$ as

$$\mathbb{I} = \sum_G \delta_G^{-1} G = \sum_{ij\eta} \frac{\ell_\eta}{k!} \Gamma^\eta_{ij} (\mathbb{I}) \Gamma^\eta_{ij} (G^{-1}) G$$

$$= \sum_{ij\eta} \frac{\ell_\eta}{k!} \Gamma^\eta_{ii} (G^{-1}) G = \sum_{\eta i} e^\eta_{ii}$$

so

$$\mathbb{I} = \sum_{\eta i} e^\eta_{ii}$$

Thus the whole algebra is spanned by these two sided ideals. In particular, the $Y_i$ are contained in the corresponding $e_{ij}^\eta$ (an $\ell_\eta^2$ dimensional algebra).

In fact, the space spanned by $e_{ij}^\eta$ is also spanned by $Q s_{ij} P_\tau$, where $Q$, and $P_\tau$ are the antisymmetrizers and symmetrizers of a set of standard tableaux for $\eta$, which means tableaux in which the numbers increase left to right in each row, and also top to bottom in each column. Thus $\boxed{1\, 2\, 3}$ and $\boxed{1\, 2\, 3}$ are standard tableaux, but $\boxed{2\, 1\, 3}$ and $\boxed{1\, 2\, 3}$ are not. Here $s_{ij}$ is the permutation such that $\tau_i = s_{ij} \tau_j$. Each of these spaces has dimension $\ell_\eta^2$, with $\ell_\eta$ equal to the number of standard tableaux, so

The dimension of $\Gamma^\eta$ is the number of standard tableaux of $\eta$.

Counting all possibilities is tedious, so we have a magic formula in terms of hooks.

For each box $b$ in a Young graph with $k$ boxes, define the hook of $b$, $g_b = 1$ plus the number of boxes directly to the right plus the number of boxes directly beneath. Then

$$\ell_\eta = \prod_b g_b$$

Example: In the Young graph I have placed the corresponding hooks (this is not a Young tableau)

$$\begin{array}{ccccc}
7 & 5 & 3 & 2 \\
6 & 4 & 2 & 1 \\
3 & 1 & & \\
& 1 & & \\
\end{array}$$

$$\ell = \frac{11!}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3^2 \cdot 2^3 \cdot 1^3} = 1320.$$
11.2 Representations of SU(n)

We now turn to the extraction of arbitrary representations of SU(n). Georgi discusses the fundamental weights of SU(n), and shows that an arbitrary representation can be found from a tensor product of an adequate number of defining representations. The problem is to extract from the tensor product of \( k \) defining representations \( \otimes \mathbb{N}^k \) the irreducible pieces. We have seen that this can be done by demanding that elements of the permutation group algebra vanish. If we impose \( e^\eta w = 0 \) for all \( \eta \) and \( i \) save one, that is equivalent to projecting out our representation

\[
T^{a_1 a_2 \cdots a_k} = (e^\eta w)^{a_1 a_2 \cdots a_k} \quad \text{no sum on } i
\]

for one particular representation \( \eta \) and one basis vector \( i \).

The different \( i \) generate equivalent representations. The different \( \eta \)'s, however, each correspond to a different (inequivalent) representation of SU(n).

Before doing more formal arguments, we will do an example. Consider three spin \( \frac{1}{2} \) objects, or the tensor product of three defining representations of SU(2). We will extract from this 8 dimensional state space the piece \( e^{11} \).

From the problem you did for homework (#3, problem 1),

\[
e^{11} = \frac{2}{6} \sum_P \Gamma^{11} \quad (P^{-1}) P
\]

\[
= \frac{1}{3} \left( \mathbb{1} + (12) - \frac{1}{2}(23) - \frac{1}{2}(13) - \frac{1}{2}(123) - \frac{1}{2}(132) \right),
\]

Let this act on the basis vectors which we expand as \( \uparrow = e_1, \downarrow = e_2 \).
We now want to extract from $S^r$ the piece projected out by $e_i^\eta$. The products $\{e_i^\eta P\}$ for all $P$ are just sums of multiples of $e_i^\eta$, for all $j$ (by p. 125) so we want to know the dimension of the space $\{e_i^\eta P\}$. As $e^\eta$ is a two-sided ideal, this space is $\sum_k e_i^\eta b_k$, so the dimensionality depends on the constraints on $b_k$. If they were all independent, they would form an $\ell_\eta$ dimensional space. But there are constraints. For $B \in \mathcal{P}$, $P_B B = P_B$. Let’s be more explicit:

$$e_i^\eta P_B = \sum_n e_m^\eta b_{nj} = e_i^\eta P_B = \sum_n b_{nj} e_m^\eta B = \sum_{nm} \Gamma_{mn}^\eta (B) e_{im} b_{nj}.$$ 

The $e_{im}^\eta$ are linearly independent, so $b_{nj} = \sum_m b_{mj} \Gamma_{mn}^\eta (B)$, for $B \in \mathcal{P}$. To find out how many degrees of freedom survive this constraint for each $j$, observe that $\Gamma_{mn}^\eta (B)$ forms a reducible representation of the subgroup $\mathcal{P}$. So we can write

$$\Gamma^\eta (B) = U \bigoplus \epsilon \Gamma^\epsilon (B) U^{-1}$$

where $\Gamma^\epsilon$ are irreducible representations of $\mathcal{P}$. Now if $c = bU$,

$$b = bU \bigoplus \epsilon \Gamma^\epsilon (B) U^{-1} \implies c = c \left( \bigoplus \epsilon \Gamma^\epsilon (B) \right).$$

The vector $c$ breaks up into pieces for each representation $\epsilon$, with $c^\epsilon \Gamma^\epsilon (B) = c^\epsilon$ for all $B \in \mathcal{P}$. This is possible for nonzero $c$ only if $\epsilon$ is the identity representation, as the representations are irreducible.

Therefore the dimensionality of the space $e_i^\eta Pe$ is the number of times, $\gamma_\eta$, that the identity representation of $\mathcal{P}$ is contained in $\Gamma^\eta$.

But the number of times the representation $i$ is contained in $\Gamma^\eta$ is

$$\gamma_\eta = a_{i=1} = \frac{1}{\gamma_\eta} \sum_{B \in \mathcal{P}} \chi^\epsilon (B) \chi^\eta (B) = \frac{1}{\gamma_\eta} \sum_{B \in \mathcal{P}} \chi^\epsilon (B)$$

for $i=$ identity, where $\gamma_\eta$ is the number of elements in $\mathcal{P}$, which is $\Pi_i r_i$.

Example: $\eta = \frac{1}{3}$. From homework, and recalling $\chi = \text{Tr } \Gamma$,

$$\chi = 2 \text{ for } B = \mathbb{I}$$

$$\chi = 0 \text{ for } B = (1 2), (1 3), \text{ or } (2 3)$$

$$\chi = -1 \text{ for } B = (1 2 3), \text{ or } (1 3 2)$$

Consider the space starting from $e_1 \otimes e_1 \otimes e_2$.

$$\mathcal{P} = \{1, (1 2)\}, \quad g_p = 2, \quad \gamma_\eta = \frac{1}{2} (2 + 0) = 1$$

so $e_{11}$ generates only one state from the three-dimensional space $S^r$.

From $e_1 \otimes e_1 \otimes e_1$, $\mathcal{P} = S_3$, $g_p = 6$, $\gamma_\eta = \frac{1}{6} (2 - 1 - 1) = 0$ so we get no state here.

If all vectors are unequal, say $e_1 \otimes e_2 \otimes e_3$ for $\text{SU}(n > 2)$, $\mathcal{P} = \mathbb{I}$, $g_p = 1$, $\gamma_\eta = \frac{2}{6} = 2$.

For $\text{SU}(N)$, there are $N$ states of the form $e_i \otimes e_j \otimes e_k$, not contributing anything to $\eta = \frac{1}{N}$. There are $N(N - 1)$ states $e_i \otimes e_j \otimes e_j$ with $i \neq j$, each contributing one state, so from these we get $N(N - 1)$ states. There are also $N(N - 1)(N - 2)/6$ states of the form $e_i \otimes e_j \otimes e_k$, with $i < j < k$, each contributing 2 states, so the dimension of $\eta$ is

$$N(N - 1) + \frac{1}{3} N(N - 1)(N - 2) = \frac{N(N^2 - 1)}{3} = \begin{cases} 2 & \text{for } N = 2 \\ 8 & \text{for } N = 3 \end{cases}$$

Let’s work another example, $\frac{1}{2}$ for $\text{SU}(N)$. As we need the characters for this representation, let’s take them from Schensted: $\chi = 3$ for $\mathbb{I} \in (1^1)$, $[1 \text{ element}]$; $\chi = 0$ for $(3, 1)$, $[8 \text{ elements}]$; $\chi = 1$ for $(2, 1^2)$, $[6 \text{ elements}]$; $\chi = -1$ for $(2, 2)$, $[3 \text{ elements}]$ and for $(4)$, $[6 \text{ elements}]$.

Enumerating the basis states in the various partitions, and multiplying $\chi(B)$ by their number for those within $\mathcal{P}$, we find

<table>
<thead>
<tr>
<th>indices</th>
<th>subspace</th>
<th>$\mathcal{P}$</th>
<th>$\gamma_\eta$</th>
<th>$\chi$ $1^4$ $3(1)$ $2(1^2)$ $(2, 2)$ $(4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>all $i$</td>
<td>$e_i e_i e_i$</td>
<td>$S_4$</td>
<td>$\frac{1}{24}$</td>
<td>$1 \cdot 3 + 8 \cdot 0 + 6 \cdot 1 + 3 \cdot (-1) + 6 \cdot (-1) = 0$</td>
</tr>
<tr>
<td>$i \neq j$</td>
<td>$e_i e_i e_j$</td>
<td>$S_3$</td>
<td>$\frac{1}{6}$</td>
<td>$1 \cdot 3 + 2 \cdot 0 + 3 \cdot 1 + 0 \cdot (-1) + 0 \cdot (-1) = 1$</td>
</tr>
<tr>
<td>$i &lt; j$</td>
<td>$e_i e_j e_j$</td>
<td>$S_2 \times S_2$</td>
<td>$\frac{1}{4}$</td>
<td>$1 \cdot 3 + 0 \cdot 0 + 2 \cdot 1 + 1 \cdot (-1) + 0 \cdot (-1) = 1$</td>
</tr>
<tr>
<td>$i \neq j &lt; k \neq i$</td>
<td>$e_i e_j e_k$</td>
<td>$S_2$</td>
<td>$\frac{1}{2}$</td>
<td>$1 \cdot 3 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot (-1) + 0 \cdot (-1) = 2$</td>
</tr>
<tr>
<td>all $i \neq j \neq k$</td>
<td>$e_i e_j e_k$</td>
<td>${1, 2, 3, 4}$</td>
<td>$\frac{1}{3}$</td>
<td>$1 \cdot 3 + 0 \cdot 0 + 0 \cdot 1 + 0 \cdot (-1) + 0 \cdot (-1) = 3$</td>
</tr>
</tbody>
</table>

Rewriting this with only the results for $\gamma_\eta$ to allow room for counting index choices and states, we have...
Consider \( r_1 = 1, r_2 = 2, r_4 = 1 \). Then we have \( \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \) and \( \begin{array}{c} \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline 1 \\ \hline \end{array} \), so again \( \gamma = 2 \), as for \( r_1 = 2, r_2 = r_3 = 1 \).

Now to count the dimensionality of an irreducible representation of \( \text{SU}(N) \) belonging to the Young graph \( \eta \), we must sum, over all choices \( r_i \), the corresponding \( \gamma \). But for each choice of \( r_i \) the \( \gamma_i \) is the number of ways of placing the indices in the graph in a permissible fashion. So the dimension of the full irreducible representation of \( \text{SU}(N) \) is the number of ways of placing \( k \) integers, chosen from \( 1, 2, \ldots, N \) (repeats allowed) in a permissible fashion in \( \eta \).

Example \( \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \) for \( \text{SU}(3) \): \( \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \), for a total of 8 dimensions.

There is an easier method of finding the dimensionality. For each box, associate the value \((N + \text{column number} - \text{row number})\). Then divide by the hook of that box. The dimension of the representation is the product of these quotients over all the boxes.

Examples: \[
\text{Dim} \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \frac{N}{N-1} \begin{array}{c} \hline 3 \\ \hline 1 \\ \hline 1 \\ \hline \end{array} = \frac{N(N^2-1)}{3}
\]

\[
\text{Dim} \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \frac{N}{N-1} \begin{array}{c} \hline 4 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} = \frac{(N+2)!}{8(N-2)!}
\]

Note: If the first column of a graph has \( N \) boxes, the hook of each box in column 1 is equal to the \((N + \text{column number} - \text{row number})\) of the last box in that row. Thus eliminating the first row does not change the dimension. In fact, it does not change the representation either. This is because a totally antisymmetric tensor with \( N \) indices is invariant.

Thus in \( \text{SU}(2) \), \( \begin{array}{c} \hline 1 \\ \hline 2 \\ \hline \end{array} \) as we saw in detail. It also means that for \( \text{SU}(N) \), we needn’t consider representations with \( N \) or more rows (except perhaps to indicate the identity representation by one column of \( N \) boxes).