Chapter 3

Infinite Groups

Most physical situations that have a symmetry group have an infinite group. Some examples:

- Rotational invariance, $SO(3)$. Here we can rotate through an arbitrary angle specified by a continuous parameter θ, restricted to some finite range, say $[0, 2\pi)$. There are a infinite continuum of possible values of θ, even though its range is limited. There are also two continuous parameters necessary to specify the direction about which this rotation is to take place. This is a three-parameter group, and the space of these transformations is a three dimensional manifold[1].

- Translational invariance of the vacuum, $\vec{x} \to \vec{x} + \vec{a}$, for $\vec{a}$ an arbitrary three dimensional vector with a continuum of possibilities for each coefficient.

- A combination of the above, $\vec{x} \to \vec{x}\,'$ with $x'_i = \sum_j R_{ij}x_j + a_i$, with R_{ij} a rotation matrix.

- Translations on a lattice which leave the lattice unchanged. A perfect lattice in D dimensions has D linearly independent **lattice vectors** $\vec{a}_i, i = 1, ..., D$, such that the lattice is unchanged if the whole thing is translated by a vector $\sum_{i=1}^{D} n_i\vec{a}_i$, where the coefficients n_i are all arbitarary *integers*.

The last example differs from the others in an essential way — there are no group elements which do arbitrarily little, although of course there is one,

[1]But not a vector space, and also it is not the space on which the rotations act.

the identity, which does nothing. For the $SO(3)$ rotations we can rotate through an arbitrarily small angle, for the translations of the vacuum we can translate by a femtometer (we theorists can — experimentalists might have a hard time). But the symmetries on the lattice have a minimum nonzero distance for which a translation can be a symmetry.

A group that has elements which are infinitesimally different from $\mathbb{1}$ is called a **continuous** group. The others are called **discrete**. A continuous group requires one or more continuous parameters to specify which element is being discussed. For example, for the translation group of the vacuum, $\vec{a} = (a_x, a_y, a_z)$ is a set of three continuous parameters needed to specify the translation. For the rotation group we can specify three Euler angles. Later we will make a better choice, but it will still require three real parameters.

Notice that we have implicitly assumed some kind of topology on the group, for we have talked of elements arbitrarily close to the identity, which implies a sequence of elements converging to the identity. For this reason these groups are also called **topological groups**.

3.1 Connectedness

With topology comes the concept of connectedness — can any two elements of the group be connected by a continuous path of elements in the group. The part of the group connected to the identity is called the **connected component**

Clearly the translations of the vacuum form a connected group, because, for any translation by $\vec{a}$, the set of translations $\{T_\lambda : \vec{x} \mapsto \vec{x} + \lambda\vec{a}$, for $\lambda \in [0, 1]\}$, is a continuous path of translations starting from the identity at $\lambda = 0$ and ending at the translation by $\vec{a}$ at $\lambda = 1$.

The proper rotations are also connected by the same approach. But if we consider the set of all transformations that preserve lengths, which is to say the set of all orthogonal transformations, $O(3)$, this includes the parity transformation $P : \vec{x} \to -\vec{x}$. It is clear, however, that there is no path of orthogonal transformations which connects this parity transformation to the identity. Parity in 3-D converts a left hand to a right hand, which can't be done continuously by orthogonal transformations. More abstractly, and in arbitrary dimension, if A is an orthogonal matrix, $A^{-1} = A^T$. Then $(\det A)^{-1} = \det(A^{-1}) = \det(A^T) = \det A$, so $\det A = \pm 1$. The identity has determinant $+1$ while parity (in 3-D) has determinant -1. But as no intermediate values are allowed for an orthogonal transformation, there is no path between them.

Thus this group, $O(3)$, consists of two pieces, the connected component, called SO(3), which is the subgroup of orthogonal matrices with determinant +1, and the piece connected to P. This second component is in fact the left coset of $SO(3)$ in $O(3)$ with respect to P, and $SO(3)$ is a normal subgroup[2].

The connected component will always form a normal subgroup, and the factor group will always be discrete. For the most part we will treat only connected groups.

The space of parameters describing the connected component of the group will form a manifold, that is, the neighborhood of each point can be described by Euclidean coordinates in n dimensions, though the metric may be only Euclidean in an infinitesimal neighborhood. We will define a metric (or measure) on the parameter space of the group later, but for now I only comment on topological issues. Besides connectedness, some other aspects of the topology of the group manifold which will come into play are whether or not it is **simply connected** and whether it is **compact**.

A manifold is simply connected if every closed path can be continuously shrunk to a point. The surface of a sphere is simply connected, the surface of a donut, or a torus, is not.

The manifold is compact if it forms a closed and bounded set in the topology we are considering. Only after defining a metric on the group manifold can we really answer the question of whether or not a sequence is a Cauchy sequence, which is necessary to define compactness. Usually the metric will be uniformly continuous in the parameters, so a sequence of elements whose parameters approach a limit themselves approach a limit. When this is true, compactness reduces to having a compact set in parameter space. But it is not always true[3].

Examples:

- $SO(2)$, the set of rotations in two dimensions[4], where $g(\theta)$ is a counterclockwise rotation through an angle $\theta \in [0, 2\pi)$. This is compact, but not simply connected. The group manifold is simply a circle.

[2] It is of index 2, see (1.6).

[3] Example: Lorentz transformations described in terms of velocity. The domain of velocity is bounded, but rapidity rather than velocity is the appropriate measure, and the space is not compact.

[4] As we are now dealing with infinite groups, we will no longer reserve g for the order of the group, but will often use it to represent an element, rather than A as we have been.

- The translations in one dimension by an arbitrary amount, $T(a) : x \to x + a$. Here $a \in (-\infty, \infty)$ and the group is not bounded and not compact. However, we can't really tell just by the range of the parameter; we might have parameterized the group differently, $T'(\theta) : x \to x + \tan(\theta/2)$, for $\theta \in (-\pi, \pi)$. This is exactly the same group of transformations as the $\{T(a)\}$, so it is still noncompact, even though the parameters are on a bounded set. We will discuss very soon why a is a more valid parameterization than θ.

- $SO(3)$, rotations in three dimensions. Recall that the group elements can be described by a 3-D real vector $\vec{\omega}$ as a rotation through an angle $|\vec{\omega}|$ about the axis in the direction of $\vec{\omega}$. But only $\vec{\omega}$'s with $|\vec{\omega}| \le \pi$ are needed, so the parameter space is a ball of radius π, and indeed the topology is strange, because for the points with $|\vec{\omega}| = \pi$, the group elements $g(\vec{\omega}) = g(-\vec{\omega})$, so opposite points of each diameter are identified, and points near the two ends of the diameter are close to each other.

3.2 Infinitesimal Generators

Let us concentrate on the connected component of the group. Suppose that the group elements, at least those sufficiently near the identity, are parameterized by a D dimensional parameter ν_i, with $g(\nu_i{=}0) = \mathbb{I}$. Any representation Γ which respects the topology of the group will then have a power series expansion

$$\Gamma(g(\nu)) = \mathbb{I} + \sum_i \nu_i \Gamma_i + \mathcal{O}(\nu_i \nu_j).$$

The D matrices Γ_i are just

$$\Gamma_i = \frac{\partial}{\partial \nu_i} \Gamma(g(\nu)) \bigg|_{\nu_j=0}.$$

Of course the Γ_i depend on the representation and each is a matrix.

More abstractly, we can consider a function f in the space of all (sufficiently differentiable) functions defined on the group. Let us define a set of differential operators L_i on this space by

$$[L_i f](g) = \frac{\partial}{\partial \nu_i} f(A(\nu) g) \bigg|_{\nu_j=0} \quad \text{for } g \in G.$$

But representations are functions on the group, and if we consider an irreducible subspace $\Gamma^k_{ab}(A)$,

$$\begin{aligned} [L_i\Gamma^k_{ab}](g) &= \left.\frac{\partial}{\partial\nu_i}\Gamma^k_{ac}(A(\nu))\right|_{\nu_j=0}\Gamma^k_{cb}(g) \\ \text{so}\quad L_i\Gamma^k_{ab} &= \Gamma^k_{ac}(L_i)\Gamma^k_{cb} \\ \text{where}\quad \Gamma^k_{ac}(L_i) &:= \left.\frac{\partial}{\partial\nu_i}\Gamma^k_{ac}(A(\nu))\right|_{\nu_j=0} \end{aligned}$$

defines a representation not of the group[5] but of the operators L_i.

Thus far we have considered only group elements in an infinitesimal neighborhood of the identity, *i.e.* $A(\nu)$ for infinitesimal ν. Let us extend this parameterization by writing, for finite ν,

$$A(\nu) = \lim_{N\to\infty}\left[A\left(\frac{\nu}{N}\right)\right]^N.$$

For any representation,

$$\Gamma(A(\nu)) = \lim_{N\to\infty}\left[\Gamma\left(A\left(\frac{\nu}{N}\right)\right)\right]^N = \lim_{N\to\infty}\left[1+\frac{\nu}{N}\Gamma_i\right]^N = \exp\sum_i \nu_i\Gamma(L_i),$$

so we can write at least formally,

$$A(\nu) = e^{\Sigma_i\nu_i L_i}.$$

It can be shown that any element in the connected component of a compact Lie group[6] is of this form, so we now have a good parameterization of the whole thing.

Example: $SO(2)$ are the rotations in two dimensions. Let us start with a very bad parameterization of these 2×2 matrices,

$$A(x) = \begin{pmatrix}\sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2}\end{pmatrix} \qquad \text{(injudicious way of expressing } A\text{)}$$

[5] Note we have two functions both called Γ^k_{ac}, one of which has group elements as its argument, and the other has the differential operators L_i (also called infinitesimal generators) as its argument. This is not *usually* confusing.

[6] `http://en.wikipedia.org/wiki/Lie_group` under "The exponential map". It says there that this is not true for $SL(2,R)$, which is not compact.

Then the one L_i is

$$L = \left.\frac{d}{dx}A(x)\right|_{x=0} = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}.$$

Note $L^2 = -\mathbb{1}$, so

$$\begin{aligned} e^{\theta L} &= \sum_n \frac{1}{n!}\theta^n L^n = \mathbb{1}\sum_{\text{even } n}\left(\frac{1}{n!}(i\theta)^n\right) - iL\sum_{\text{odd } n}\frac{1}{n!}(i\theta)^n \\ &= \mathbb{1}\cos\theta + L\sin\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}. \end{aligned}$$

So although we started with a deliberately poor parameterization in terms of x, we find a natural parameterization in terms of θ. And any rotation in $SO(2)$ can be written as $A(\theta) = e^{\theta L}$. L is the only generator of $SO(2)$.

Clearly $A(\theta_1)A(\theta_2) = A(\theta_1+\theta_2)$ so the group is Abelian.

All of the irreducible representations of any Abelian group are one dimensional, because all the representatives commute with each other and therefore can be simultaneously diagonalized. Thus for our SO(2),

$$\Gamma^{(m)}(A(\theta)) = e^{im\theta} = \chi^{(m)}(A(\theta)).$$

As $\theta + 2\pi$ describes the same group element as θ, $A(2\pi) = \mathbb{1}$ and we must[7] have $\Gamma(A(2\pi)) = e^{2\pi i m} = 1$, so m must be an integer. We have a countable infinity of representations.

The group-invariant volume for this group is just $d\theta$, which is left invariant under left multiplication by $A(\phi)$ because

$$\begin{aligned} \int_0^{2\pi} d\theta\, f(A(\phi)A(\theta)) &= \int_0^{2\pi} d\theta\, f(A(\theta+\phi)) = \int_\phi^{\phi+2\pi} d\theta'\, f(A(\theta')) \\ &= \int_0^{2\pi} d\theta\, f(A(\theta)). \end{aligned}$$

[7] Of course in quantum mechanics we will consider fermions, which are not actually representations of the rotations group $SO(3)$ because under a rotation by 2π, the wave function changes sign. Fermions are actually representations of the covering group $SU(2)$ of $SO(3)$, under which one must rotate by 4π to get back to the identity. Then we find that m must be half of an integer.

So we expect orthogonality of the characters in the continuous version:

$$\int_0^{2\pi} \chi^{m*}(\theta)\chi^n(\theta)\,d\theta = 2\pi\delta_{mn}.$$

The functions $e_m(\theta) = e^{im\theta}$, for $m = -\infty ... \infty$ form a complete set of functions on the group, which as a set is just a circle.

$SO(2)$ is clearly connected. It is clearly multiply-, not simply-, connected, because the path $\lambda \to A(2\pi n\lambda)$ for $\lambda \in [0,1]$ is a closed path which cannot be continuously deformed to a point because it wraps around the circle n times.

We have claimed that the group elements for the connected component of any compact Lie group can all be written as

$$g(\nu) = e^{\Sigma_i \nu_i L_i}$$

where the L_i are the generators of the group. The number of independent L_i's is called the dimension of the group[8].

How does the multiplication law of the group manifest itself in properties of the generators L_i? For small ν_1, ν_2,

$$g(\nu_1)g(\nu_2) \sim (1+\sum_i \nu_{1i}L_i)(1+\sum_j \nu_{2j}L_j) = 1 + \sum_i(\nu_{1i}+\nu_{2i})L_i + \mathcal{O}(\nu_1\nu_2),$$

so a great deal of the group multiplication is built into our choice of parameters, where it is reflected additively. To see more we need to go to higher order:

$$\begin{aligned} g(\nu_1)g(\nu_2) &= 1 + \sum_i(\nu_{1i}+\nu_{2i})L_i + \sum_{ij}\nu_{1i}\nu_{2j}L_iL_j + \mathcal{O}(\nu_1^2,\nu_2^2), \\ g(\nu_2)g(\nu_1) &= 1 + \sum_i(\nu_{1i}+\nu_{2i})L_i + \sum_{ij}\nu_{2j}\nu_{1i}L_jL_i + \mathcal{O}(\nu_1^2,\nu_2^2). \end{aligned}$$

We see that if the group is Abelian, $g(\nu_1)g(\nu_2) = g(\nu_2)g(\nu_1)$, then $[L_i, L_j] = 0$, and the generators commute. If the generators do all commute, then

$$e^{\Sigma_i\nu_{1i}L_i}e^{\Sigma_i\nu_{2i}L_i} = e^{\Sigma_i(\nu_{1i}+\nu_{2i})L_i} = e^{\Sigma_i\nu_{2i}L_i}e^{\Sigma_i\nu_{1i}L_i}, \qquad \text{(Abelian group)}$$

[8]It is also the dimensionality of the manifold, or of the tangent space at the identity, and also of the tangent space everywhere else.

and the group multiplication simply corresponds to addition in the vector space spanned by the L_i's, and the statement that the group elements commute is true, not just perturbatively.

If the generators do not commute, then the multiplication

$$e^{\Sigma_i\nu_{1i}L_i}e^{\Sigma_i\nu_{2i}L_i} = e^{\Sigma_i\nu_{3i}L_i}$$

will have an expression for $\nu_3(\nu_1,\nu_2)$ which is a more complicated function of its arguments. We may, however, expand ν_3 in a power series in ν_1 and ν_2, and as we saw above it begins with $\nu_3 = \nu_1 + \nu_2 + ...$ Expanding to second order, (summations understood[9])

$$\begin{aligned} g(\nu_1)g(\nu_2) &= \left(1+\nu_{1i}L_i + \frac{1}{2}\nu_{1i}\nu_{1j}L_iL_j\right)\left(1+\nu_{2i}L_i+\frac{1}{2}\nu_{2i}\nu_{2j}L_iL_j\right) \\ &= \left(1+\nu_{3i}L_i+\frac{1}{2}(\nu_{1i}+\nu_{2i})(\nu_{1j}+\nu_{2j})L_iL_j\right), \end{aligned}$$

where in the last term, which is quadratic in ν_3, the first order expression for $\nu_3(\nu_1,\nu_2)$ is sufficient. Expanding the two sides we have

$$\begin{aligned} & 1+(\nu_{1i}+\nu_{2i})L_i + \left(\frac{1}{2}\nu_{1i}\nu_{1j}+\frac{1}{2}\nu_{2i}\nu_{2j}+\nu_{1i}\nu_{2j}\right)L_iL_j \\ = {} & 1+\nu_{3i}L_i+\left(\frac{1}{2}\nu_{1i}\nu_{1j}+\frac{1}{2}\nu_{2i}\nu_{2j}+\frac{1}{2}\nu_{1i}\nu_{2j}+\frac{1}{2}\nu_{2i}\nu_{1j}\right)L_iL_j \end{aligned}$$

Subtracting gives $0 = (\nu_{3i}-\nu_{1i}-\nu_{2i})L_i - \frac{1}{2}\nu_{1i}\nu_{2j}[L_i,L_j]$. As the L_i are a complete set, this can only have a solution for ν_3, as it must, if

$$[L_i, L_j] = c_{ij}{}^k L_k$$

for some set of coefficients $c_{ij}{}^k$, which are called the **structure constants** of the group. Clearly a group is Abelian if and only if all the structure constants are zero.

We see that the generators of the group form a Lie[10] Algebra.

Definition: An r dimensional **Lie Algebra** $\mathcal{L}$ is an r dimensional vector space together with a bilinear composition $[\cdot,\cdot] : \mathcal{L}\times\mathcal{L}\to\mathcal{L}$ with the properties

$$[x,y] = -[y,x]$$

[9]Students who are not fully expert at using indices without making mistakes should read "On Indices and Arguments" on the Supplementary Notes webpage.

[10]Marius Sophus Lie 1842–1899

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$

The second equation is called the **Jacobi identity**. These requirements are automatically satisfied if the $[\cdot,\cdot]$ law is defined as the commutator of an associative multiplication law, $[x,y]=xy-yx$. But the commutator itself is not associative,

$$[x,[y,z]]-[[x,y],z]=[x,[y,z]]+[z,[x,y]]=-[y,[z,x]]\neq 0.$$

Note that the two laws of $[\cdot,\cdot]$ imply

$$c_{ij}{}^{k}=-c_{ji}{}^{k},$$

$$c_{i\ell}{}^{m}c_{jk}{}^{\ell}+c_{j\ell}{}^{m}c_{ki}{}^{\ell}+c_{k\ell}{}^{m}c_{ij}{}^{\ell}=0.$$

Example 1: $SO(2)$

For $SO(2)$ there is only one generator, and by antisymmetry $c_{11}{}^{1}=0$, and the group is Abelian.

Example 2: $SO(3)$

$SO(3)$ is the group of rotations in three dimensions. Consider a rotation about the z-axis, (viewed as an active transformation $\vec{r}\to\vec{r}\,'$) :

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned} \quad \text{so} \quad \begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

The infinitesimal generator is therefore

$$L_z=\frac{d}{d\theta}R_z(\theta)=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Similarly,

$$L_x=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} \qquad L_y=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}$$

To calculate the structure constants we expand

$$\begin{aligned}
[L_x,L_y] &= \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}=L_z\\
[L_y,L_z] &= \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}-\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}=L_x\\
[L_x,L_z] &= \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}-\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}=-L_y
\end{aligned}$$

This should be familiar to you except for some i's. This is because, for the moment, I am using mathematician's notation for the generators. Physicists like to think of the group elements as unitary operators but the generators as hermitian, so we write

$$U=e^{-i\Sigma_j\theta_j L_j^P}$$

with Physicist's generators

$$L_j^P=iL_j, \qquad L_j=-iL_j^P,$$

so

$$[L_x^P,L_y^P]=iL_z^P, \qquad etc., \qquad \text{or better:}\quad [L_j^P,L_k^P]=i\epsilon_{jk\ell}L_\ell^P$$

which should be more familiar[11]. We also see for a Lie algebra in general that

$$[L_j^P,L_\ell^P]=ic_{j\ell}{}^{k}L_k^P$$

with the same structure constants $c_{j\ell}{}^{k}$ the mathematicians use.

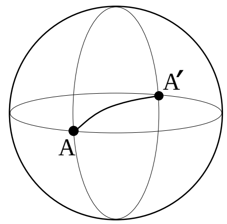
We see that for $SO(3)$, $c_{jk}{}^{\ell}=\epsilon_{jk\ell}$. Note that a rotation through angle θ about a general axis $\hat\omega$, (with $\hat\omega^2=1$) is given by $e^{-i\theta\hat\omega_j L_j^P}$. Then $\vec\omega=\theta\hat\omega$ can be used as the parameters for the group, $g(\vec\omega)=e^{-i\vec\omega\cdot\vec L^P}$, and the space of these parameters is a ball in three dimensions, $|\vec\omega|\le\pi$.

Note that for any given axis, a rotation through π is the same transformation as a rotation about the same axis through $-\pi$. This means that the

[11] If you don't know all about $\epsilon_{jk\ell}$ and how to use it in calculations, see "ϵ_{ijk} and cross products in 3-D Euclidean space" on the Supplementary Notes webpage.

group manifold is the closed ball $|\vec{\omega}| \leq \pi$, but with the opposite ends of each diameter of the ball identified with each other.

Then the path shown is a closed path, because its ends, the points A and A' at the opposite ends of a diameter, are considered to be the same point. No matter how we try to continuously deform this path, the endpoints always stay opposite each other, and we cannot shrink the path to a point. Thus the manifold of $SO(3)$ is not simply connected.



In fact, the path AA' above can be continuously deformed into any other diameter, so any path on the $SO(3)$ manifold is deformable either into a point or into a particular diameter. In fact, AA' can be deformed into its "negative", the path taken in the reverse direction. Thus the path given by adding AA' to itself, in the sense of gluing the tail of the first path to the head of the second, is a closed path which **is** deformable to the identity[12].

Example 3: $SU(2)$

The group $U(N)$ is the set of unitary $N \times N$ matrices under ordinary matrix multiplication. As for $O(N)$, the group of $N \times N$ real orthogonal matrices, it is useful to limit ourselves to those with determinant equal to 1, called $SU(N)$ and $SO(N)$ respectively. So $SU(2)$ is the group of 2×2 complex unitary matrices with determinant 1. A unitary matrix U can always be written $U = e^{iH}$ with H a hermitian matrix (proof: diagonalize first, prove it for the diagonalized version, then observe that the similarity transformation factors out). We can also use the useful formula

$$\det U = e^{\operatorname{Tr} \ln U}$$

so $\det U = e^{i \operatorname{Tr} H} = 1$ implies[13] $\operatorname{Tr} H = 0$, so H is hermitian and traceless and is therefore a real linear combination of the Pauli matrices σ_j, $H = \frac{1}{2}\sum_j \omega_j \sigma_j$.

[12]Two paths are homotopic if they can be continuously deformed into one another, so simply-connected means all closed paths are homotopic to a point (a path that doesn't move). For $SO(3)$, we see that all paths are either homotopic to a point or to a given diameter. Homotopy defines an equivalence relation on the set of closed paths, and this gives a group called the first homotopy group $\pi_1(\mathcal{M})$ on any manifold $\mathcal{M}$. Thus $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

[13]Well, at least for small H. We are looking for the infinitesimal generators, so that is sufficient.

We have already parameterized the group in terms of its generators

$$L_j^P = \frac{1}{2}\sigma_j.$$

The factor of $\frac{1}{2}$ is conventional, so that the L_i's are normalized so as to have the same structure constants as for SO(3). For

$$[L_j^P, L_\ell^P] = \frac{1}{4}[\sigma_j, \sigma_\ell] = \frac{i}{2}\epsilon_{j\ell k}\sigma_k = i\epsilon_{j\ell k}L_k,$$

so the structure constants are

$$c_{j\ell}{}^{k} = \epsilon_{j\ell k} \quad \text{just as for } SO(3).$$

Thus the Lie algebra of $SU(2)$ and the Lie algebra of $SO(3)$ are the same. The groups are **locally isomorphic**. But a "rotation" through θ about the j axis gives

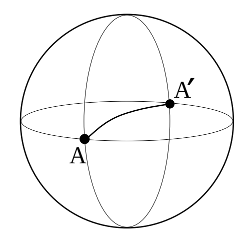
$$\begin{aligned} e^{i\theta\sigma_j/2} &= \sum_n \frac{(-1)^n}{2n!}\left(\frac{\theta}{2}\right)^{2n} + i\sigma_j \sum_n \frac{1}{(2n+1)!}(-1)^n \left(\frac{\theta}{2}\right)^{2n+1} \\ &= \cos(\theta/2) + i\sigma_j \sin(\theta/2), \end{aligned}$$

so $\theta = \pm\pi$ gives $e^{\pm i\pi\sigma_j/2} = \pm i\sigma_j$, which are not the same.

In fact, the group space now consists of all $|\vec{\omega}| \leq 2\pi$ rather than $|\vec{\omega}| \leq \pi$, but on the boundary $|\vec{\omega}| = 2\pi$

$$e^{-i\vec{\omega}\cdot\vec{\sigma}/2} = \cos(\omega/2) - i\vec{\omega}\cdot\vec{\sigma}\sin(\omega/2) \underset{|\omega|\to 2\pi}{\longrightarrow} -1,$$

so all the points on the surface of the ball, $|\omega| = 2\pi$ are identified. And we now have a simply connected manifold, because if we consider the path AA' now, we can deform it by moving A' without moving A, because all the points on the surface are the same, and so we can bring A' back to A and then shrink the rest of the path to a point.



Thus $SU(2)$ is simply connected. It is said to be the **covering group** of $SO(3)$. The subgroup $\{\mathbb{1}, -\mathbb{1}\}$ is obviously a normal subgroup $\mathbb{Z}_2$ of $SU(2)$, and $SU(2)/Z_2 \cong SO(3)$. Every point in $SO(3)$ corresponds to two points in $SU(2)$. Every representation Γ of $SU(2)$ thus provides two matrices for each element of $SO(3)$, and the product of one of these for A and one for B will give one of the two matrices for AB, but in general there is no way to select, for each element $g \in SO(3)$, a unique choice $\Gamma^j(g)$ such that the product for two elements will always give the correct choice for the product. Instead, we have

$$\Gamma(g_1)\Gamma(g_2) = \pm\Gamma(g_1 g_2).$$

This is familiar from quantum mechanics. Representations of the rotation group with $j = \frac{2n+1}{2}$ are not really representations at all, because they change sign under rotation by 2π. These representations are used by fermions, and we escape the ill-definedness of the representation by insisting that only quadratic expressions in the fermions have physical meaning.

Most of our understanding of Lie groups comes from studying the Lie algebra of the generators. The study of these will permit us to find the representations, and to classify all finite dimensional compact simply connected Lie groups. We will do so following the book by Georgi[14].

3.3 Adjoint Representation, Killing Form, *etc.*

We will assume our algebra is finite dimensional and over the reals. From now on we will use Physicist's generators, so

$$g = e^{i\omega^i L_i}, \qquad [L_i, L_j] = i c_{ij}{}^k L_k.$$

Every finite dimensional Lie group has an **adjoint representation**, given by

$$\Gamma^{\rm adj}_{jk}(L_i) = i c_{ji}{}^k.$$

Note
$$\begin{aligned} \Gamma^{\rm adj}_{ab}\left([L_i, L_j]\right) &= i c_{ij}{}^k \Gamma^{\rm adj}_{ab}\left(L_k\right) = -c_{ij}{}^k c_{ak}{}^b \\ &= c_{ai}{}^k c_{jk}{}^b + c_{ja}{}^k c_{ik}{}^b = \left(i c_{ai}{}^k\right)\left(i c_{kj}{}^b\right) - \left(i c_{aj}{}^k\right)\left(i c_{ki}{}^b\right) \\ &= \Gamma^{\rm adj}_{ak}\left(L_i\right)\Gamma^{\rm adj}_{kb}\left(L_j\right) - \Gamma^{\rm adj}_{ak}\left(L_j\right)\Gamma^{\rm adj}_{kb}\left(L_i\right) \\ &= \left[\Gamma^{\rm adj}\left(L_i\right), \Gamma^{\rm adj}\left(L_j\right)\right]_{ab}, \end{aligned}$$

[14]Howard Georgi, "Lie Algebras in Particle Physics" Second Edition, Westview Press (1999). Note the second edition is a considerable expansion of the first (Addison-Wesley, 1982)

verifying that it is a representation of the Lie algebra.

The adjoint representation has the same dimension as the algebra. It is used to construct a bilinear form on the algebra $\beta : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ given by its action on the generators

$$\beta(L_i, L_j) = \mathrm{Tr}\left(\Gamma^{\rm adj}\left(L_i\right)\Gamma^{\rm adj}\left(L_j\right)\right) = -c_{ai}{}^b c_{bj}{}^a = \beta_{ij}.$$

Note that although we have traced the product of the Γ's, we still have the indices i and j left over, and in these $\beta(L_i, L_j)$ is a symmetric real matrix, which can be diagonalized[15]. Doing so corresponds simply to a change in basis L_i of the vector space $\mathcal{L}$. So $\beta(L_i, L_j) = k_i \delta_{ij}$ in this basis.

Furthermore, by changing the scale of the basis vectors L_i, we can change the magnitude of k_i. But we cannot change the sign or whether or not it is zero. We could, however, normalize our L_i so that each k_i is ± 1 or 0.

The form β is called the **Killing form**[16]. The singularity of the matrix β (the existence of $k_i = 0$) is tied up with whether or not there is an abelian invariant subalgebra.

An **ideal** or **invariant subalgebra** $\mathcal{H}$ of $\mathcal{L}$ is a subspace such that $[\mathcal{H}, \mathcal{L}] \in \mathcal{H}$, that is, $\forall h \in \mathcal{H}, \forall \ell \in \mathcal{L}$, we have $[h, \ell] \in \mathcal{H}$. An invariant subalgebra generates a *normal subgroup*. $\mathcal{H}$ is abelian if $\forall h_1, h_2 \in \mathcal{H}, [h_1, h_2] = 0$.

If an algebra has no nontrivial invariant subalgebra it is called **simple**[17]. Here trivial means either the whole algebra $\mathcal{L}$ or the algebra $\{0\}$ consisting only of the zero element.

If an algebra has no nontrivial abelian invariant subalgebra it is **semisimple**.

Theorem: β is a singular matrix if and only if $\mathcal{L}$ is not semisimple.

Theorem: $\mathcal{L}$ is semisimple if and only if it is the direct sum of simple ideals[18].

[15]Real symmetric matrices are diagonalizable by an orthogonal matrix.

[16]The mathematician Jacobson in his book "Lie Algebras" defines β with mathematician's generators, so his is -1 times ours. That means for him β is negative definite for a compact group, as we shall see.
PS: Wilhelm Karl Joseph Killing, 1847-1923. That it kills an abelian invariant subalgebra is just a plus.

[17]Compare to simple finite groups, which means they have not nontrivial normal subgroup.

[18]Jacobson: *Lie Algebras* p. 71.

Example 1: SO(3) or SU(2):

$$\begin{aligned} c_{ij}^{\ \ k} &= \epsilon_{ijk} \\ \beta_{ij} &= -\sum_{ab} \epsilon_{aib}\epsilon_{bja} = 2\delta_{ij} \end{aligned}$$

which is already diagonalized and all k_i normalized to be equal (although not to 1). This is a nonsingular matrix, so the algebra is semisimple. In fact, as any rotational direction can be rotated into any other, it is simple.

Example 2: The Poincaré group:

For a relativistic system, Physics is unchanged by translations and Lorentz transformations[19]

$$x^\mu \to x'^{\,\mu} = \sum_\nu a^\mu{}_\nu x^\nu + b^\mu, \qquad a, b\,\text{constants.}$$

The matrix a is constrained to be a Lorentz transformation preserving

$$(dx^0)^2 - (d\vec{x})^2 = \sum_{\mu\nu} \eta_{\mu\nu}\, dx^\mu dx^\nu, \quad \text{where} \quad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

η is called the **Minkowski metric**. The group of such transformations is called the **Poincaré group**. The **Lorentz group** is the subgroup for which $b = 0$. The condition on a is a pseudo-orthogonality condition, in that $dx'^{\,\mu} = \sum_\nu a^\mu{}_\nu dx^\nu$, so $\sum_{\mu\rho} \eta_{\mu\rho} dx'^{\,\mu} dx'^{\,\rho} = \sum_{\mu\nu\rho\sigma} \eta_{\mu\rho} a^\mu{}_\nu dx^\nu a^\rho{}_\sigma dx^\sigma = \sum_{\rho\sigma} \eta_{\rho\sigma} dx^\rho dx^\sigma$ only if $\sum_{\mu\rho} \eta_{\mu\rho} a^\mu{}_\nu a^\rho{}_\sigma = \eta_{\nu\sigma}$. If $\eta_{\mu\rho}$ were $\delta_{\mu\rho}$, this would be the condition for orthogonality of the matrix $a^{\cdot}{}_{\cdot}$. Because of the -1's we say a is **pseudo-orthogonal**.

For orthogonal matrices we can write $\mathcal{O} = e^G$, where G is an antisymmetric real matrix, $G = -G^T$, so we suspect to get pseudo-orthogonality we need $a = e^G$ with $\eta G = -G^T \eta$, *i.e.* $\sum_\rho \eta_{\mu\rho} G^\rho{}_\nu = -\sum_\sigma G^\sigma{}_\mu \eta_{\sigma\nu}$.

We can check this by noting $\eta G^2 = -G^T \eta G = (G^T)^2 \eta$, and similarly for any function of G, $\eta f(G) = f(-G^T)\eta$, so with f the exponential function, $\eta\mathcal{O} = (\mathcal{O}^{-1})^T \eta$, *i.e.* $\sum_\rho \eta_{\mu\rho} a^\rho{}_\nu = \sum_\sigma (a^{-1})^\sigma{}_\mu \eta_{\sigma\nu}$ or

$$\sum_{\mu\rho} \eta_{\mu\rho} a^\rho{}_\nu a^\mu{}_\tau = \eta_{\tau\nu} \qquad \text{as required.}$$

[19] $x^0 := ct$. Those people who don't know why some indices are up and some are down should, for now, ignore that fact.

Thus the generators consist of the four momenta P_μ with

$$\left(e^{i\Sigma_\mu b^\mu P_\mu}\, x\right)^\nu = x^\nu + b^\nu,$$

and the six Lorentz transformation generators $L^\mu{}_\nu$ with

$$\sum_\rho \eta_{\mu\rho} L^\rho{}_\nu = -\sum_\rho \eta_{\nu\rho} L^\rho{}_\mu, \quad \text{with} \quad \left(e^{i\sum_{\mu\nu} G^\mu{}_\nu L^\nu{}_\mu/2}\, x\right)^\rho = \sum_\sigma a^\rho{}_\sigma x^\sigma.$$

$L^\nu{}_\mu$ acts like $-ix^\nu\partial_\mu + ix_\mu\partial^\nu$, or better, let us define[20] $L_{\mu\nu} = \sum_\rho \eta_{\mu\rho} L^\rho{}_\nu$. Then $L_{\mu\nu}$ acts like $-ix_\nu\partial_\mu + ix_\mu\partial_\nu$, and we also have P_μ acts like $-i\partial_\mu$. Then

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\sigma}] &= ((i\eta_{\rho\mu} L_{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma)) \\ &= i\eta_{\rho\mu} L_{\nu\sigma} - i\eta_{\rho\nu} L_{\mu\sigma} - i\eta_{\sigma\mu} L_{\nu\rho} + i\eta_{\sigma\nu} L_{\mu\rho} \\ [L_{\mu\nu}, P_\rho] &= -i\eta_{\rho\nu} P_\mu - (\mu \leftrightarrow \nu) = -i\eta_{\rho\nu} P_\mu + i\eta_{\rho\mu} P_\nu \\ [P_\mu, P_\nu] &= 0. \end{aligned}$$

The four dimensional algebra generated by the P's is invariant and abelian, so the Poincaré group is not semisimple. The six dimensional algebra generated by the L's is a subalgebra but not invariant. Considered by itself, it is called the **Lorentz algebra** and generates the Lorentz group. It is semisimple, but it is not compact. Considering just Lorentz boosts in one dimension, the appropriate (measure-preserving) parameter is not velocity but rapidity ϕ, with $\beta = v/c = \tanh\phi$. But then the range of the parameter ϕ is $(-\infty, \infty)$ and is not compact. If you look at the diagonalized Killing form for the Lorentz algebra, you find k_i's of both signs. There is a theorem that says all the k_i's are positive if and only if the group is compact. Compact semisimplicity also means the irreducible unitary representations are finite dimensional. This is not true for the Lorentz group, for which finite dimensional representations are not unitary. That is why we need $\bar\psi$ instead of $\psi^\dagger$ for fermion fields in quantum field theory.

3.4 Quantum Operators

Let us return to compact semisimple algebras generating symmetry groups which do have unitary representations.

[20] From now on, we cannot ignore whether an index is up or down — the two quantities, called contra- and co-variant, are related with the Minkowski metric.

The states of a physical system having a symmetry group G transform under symmetry transformations according to some representation. That is, we can find a basis of states $|i\rangle$ and the operators of the group are unitary operators on the states,

$$|i\rangle \to e^{i\vec{\nu}\cdot\vec{L}}\,|i\rangle = \sum_j |j\rangle\,\Gamma_{ji}\big(e^{i\vec{\nu}\cdot\vec{L}}\big).$$

Bras are the hermitian conjugates. Assuming unitarity,

$$\langle i| \to \langle i|\,e^{-i\vec{\nu}\cdot\vec{L}} = \sum_j \Gamma_{ij}\big(e^{-i\vec{\nu}\cdot\vec{L}}\big)\,\langle j|$$

as Γ is unitary.

$$\begin{aligned} \text{Note}\quad \langle k|\ell\rangle \;&\to\; \sum_{jm} \Gamma_{kj}\big(e^{-i\vec{\nu}\cdot\vec{L}}\big)\,\langle j|m\rangle\,\Gamma_{m\ell}\big(e^{i\vec{\nu}\cdot\vec{L}}\big) \\ &= \sum_{jm} \Gamma_{kj}\big(e^{-i\vec{\nu}\cdot\vec{L}}\big)\,\delta_{jm}\,\Gamma_{m\ell}\big(e^{i\vec{\nu}\cdot\vec{L}}\big) = \Gamma_{k\ell}(\mathbb{1}) = \delta_{k\ell}. \end{aligned}$$

so the scalar products are invariant under the action of the group, for unitary representations.

Consider an operator $\mathcal{O}$ which corresponds to a physical variable p. If I measure p in a state ψ, I get various values averaging to $p = \langle\psi|\,\mathcal{O}\,|\psi\rangle$.

If I ask how a physical variable is changed under the action of a symmetry transformation, I measure the new value p' by inserting $\mathcal{O}$ between the transformed states

$$p' = \langle\psi'|\,\mathcal{O}\,|\psi'\rangle = \langle\psi|\,e^{-i\vec{\nu}\cdot\vec{L}}\mathcal{O}e^{i\vec{\nu}\cdot\vec{L}}\,|\psi\rangle\,.$$

Thus I can equivalently think of the transformation as acting on the operator and leaving the states alone:

$$\mathcal{O} \to \mathcal{O}' = e^{-i\vec{\nu}\cdot\vec{L}}\mathcal{O}e^{i\vec{\nu}\cdot\vec{L}}.$$

Under an infinitesimal transformation ν,

$$\delta\mathcal{O} = \mathcal{O}' - \mathcal{O} = -i\nu_a\,[L_a, \mathcal{O}]\,.$$

Note my interpretation of the symmetry action on $\mathcal{O}$ is opposite to Georgi. His is a kind of compensating transformation, while mine is active.

There is some advantage to thinking of the symmetry operations as passive, a kind of compensating transformation, as Georgi does. This is especially useful for rotations, as the symmetry transformation can be thought of as not affecting the state at all, but only its description in terms of a rotated coordinate systems. Suppose ψ and $\mathcal{O}$ are a wave function and an operator described with respect to an original coordinate system, and ψ' and $\mathcal{O}'$ the same state and operator described in a rotated one. Then the eigenvalue $\langle\psi|\,\mathcal{O}\,|\psi\rangle$ must change only as appropriate for an object of $\mathcal{O}$'s spin. This leads to very powerful constraints on the matrix elements, known as the Wigner-Eckart theorem.

First we will work out the needed mathematics for SU(2), and then consider an arbitrary semisimple compact Lie group. The things we must discuss are

- Irreducible representations of the group
- Completeness and orthogonality
- The Wigner-Eckart theorem

There are several classic standard reference texts on SU(2):

- Condon and Shortley "Theory of Atomic Spectra"
- Edmunds "Angular Momentum in Quantum Mechanics" QC174.E4
- Rose "Elementary Theory of Angular Momentum" QC174.1R7