Chapter 14

Hamiltonian Formulation of Local Symmetry

For several purposes, it is sometimes preferable to work on a lattice with different lattice spacings in different directions, as we have done in analyzing the plaquette. We can write the action as a sum over sites with the appropriate space-time volume,

$$\begin{aligned}
\int d^4x &\rightarrow a^x a^y a^z a^t \sum_n \\
U_{n,n+\mu} &= e^{iga^\mu \mathcal{A}_\mu} \qquad \text{on each link} \\
F_{\mu\nu}^2 &\rightarrow -\sum_{\mu\nu} \frac{1}{g^2 a^{\mu\,2} a^{\nu\,2}} \operatorname{Tr}\left(U_{p_{\mu\nu}} + U^\dagger_{p_{\mu\nu}} - 2\cdot \mathbb{1}\right) \qquad \text{for each placquette}
\end{aligned}$$

and so[1]

$$\begin{aligned}
S &= a^x a^y a^z a^t \sum_n \left\{ \frac{1}{4g^2} \sum_{\mu\nu} \frac{1}{a^{\mu\,2} a^{\nu\,2}} \operatorname{Tr}\left(U_{p_{\mu\nu}} + U^\dagger_{p_{\mu\nu}} - 2\cdot \mathbb{1}\right) (-1)^{\delta_{\mu 0} + \delta_{\nu 0}} \right. \\
&\qquad \left. + i \sum_\mu \frac{1}{a^\mu} \bar{\psi}_{n+\mu} \gamma^\mu U_{n,n+\mu} \psi_n - m\bar{\psi}\psi \right\}.
\end{aligned}$$

We now turn to a discussion of the Hamiltonian formulation, as well as the Lagrangian. This will give us a quantum mechanics in the usual language,

[1] I am leaving out a discussion of fermion doubling. In general we shall concentrate on the gauge field U.

but of a rather unusual form, because the kinetic energy term will not be the usual $\sum p_i^2/2m$, because the degrees of freedom, the group elements on each link, do not live in a Euclidean space, but instead live on the group manifold. The way the derivative operators enter the Hamiltonian will lead us to a metric and a measure on the group manifold.

We will derive the Hamiltonian formulation of the lattice gauge theory by considering the $a^t \to 0$ limit. First we must use gauge invariance to remove some of the degrees of freedom.

Quantum mechanics is an integral of the action over all possible configurations of the physical degrees of freedom. Gauge degrees of freedom have no effect on the action, and are therefore not physical degrees of freedom. Given any configuration U, ψ on all the links and sites, it is possible to find a new configuration, gauge transformed, satisfying some imposed gauge requirements but representing the same physical configuration. In our case, we pick a particular time, say $t = 0$. We now perform gauge transformations at each site with $t \neq 0$ in order to make all $U_{n,n+\hat{t}} = 1\!\!\mathrm{I}$. This can be done iteratively. For example, at $t = a^t$, at site $n = \vec{r}, a^t$ we examine $U_{(\vec{r},0),\hat{t}} = e^{i\omega_{\vec{r}}}$, and then make a gauge transformation at the site $(\vec{r}, a^t)$ with $\omega_{\vec{r}}$, with no change at $(\vec{r}, 0)$, so the new U on the timelike link is

$$U'_{(\vec{r},0),\hat{t}} = e^{-i\omega_{\vec{r}}} e^{i\omega_{\vec{r}}} e^0 = 1\!\!\mathrm{I}.$$

This set of transformations changes all the spacelike U's at $t = a^t$ to new values, but it does not affect the integral over all such U's, provided the weight of integration is group invariant. We shall say more about this later.

Reiterating this step forwards in time to $t = \infty$, and also going backwards from $t = 0$, we wind up with a configuration of U's and ψ's equivalent to our original one, but subject to the gauge constraint $U_{n,\hat{t}} = 1\!\!\mathrm{I}$. In the continuum limit, this means $A_0 = 0$ everywhere, which is called the **temporal gauge**.

Now let us consider the limit $a^t \to 0$ with $a^x = a^y = a^z = a$ held fixed. The sum over times, times a^t, is just $\int dt$, so $S = \int dtL$ as usual, with

$$\begin{aligned} L &= \sum_{n_x,n_y,n_z} a^3 \Bigg\{ \frac{-1}{2g^2 a^2 a^{t\,2}} \sum_j \left(\mathrm{Tr}\, U_{P_{0j}} - 1\!\!\mathrm{I} + \mathrm{h.c.} \right) \\ &\quad + \frac{1}{4g^2 a^4} \sum_{ij} \mathrm{Tr} \left(U_{p_{ij}} + U^\dagger_{p_{ij}} - 2 \cdot 1\!\!\mathrm{I} \right) + \text{fermion terms.} \end{aligned}$$

The first term is the sum over placquettes in a time-space plane, while the second is in a space-space plane (at a fixed time). The first of these involves

$$U_{p_{0j}} = \underbrace{U^\dagger_{n^\nu,0}}_{1\!\!1} U^\dagger_{n^\nu+a^t,j} \underbrace{U_{n^\nu+a^j,0}}_{1\!\!1} U_{n,j}.$$

Expand

$$\begin{aligned} U_{(\vec n,t+a^t),j} &= U_{(\vec n,t),j} + a^t \frac{\partial U_{(\vec n,t),j}}{\partial t} + \frac{1}{2} a^{t\,2} \frac{\partial^2 U}{\partial t^2}, \\ \text{so} \qquad U^\dagger_{(\vec n,t+a^t),j} U_{(\vec n,t),j} - 1\!\!1 &= a^t \left(\frac{\partial U^{-1}_{n^\nu,j}}{\partial t} \right) U_{n^\nu,j} + \frac{a^{t\,2}}{2} \frac{\partial^2 U^{-1}_{n^\nu,j}}{\partial t^2} U_{n^\nu,j}. \end{aligned}$$

Adding the hermitean conjugate gives

$$a^t \left(\frac{\partial U^{-1}_{n^\nu,j}}{\partial t} U_{n^\nu,j} + U^{-1}_{n^\nu,j} \frac{\partial U_{n^\nu,j}}{\partial t} \right) + \frac{a^{t\,2}}{2} \left(\frac{\partial^2 U^{-1}}{\partial t^2} U + U^{-1} \frac{\partial^2 U}{\partial t^2} \right).$$

The first term is $a^t \frac{\partial}{\partial t} \left(U^{-1} U \right) = a^t \frac{\partial}{\partial t} 1\!\!1 = 0$, while from the second we may subtract

$$\frac{a^{t\,2}}{2} \frac{\partial}{\partial t} 0 = \frac{a^{t\,2}}{2} \frac{\partial}{\partial t} \left(\frac{\partial U^{-1}}{\partial t} U + U^{-1} \frac{\partial U}{\partial t} \right) = \frac{a^{t\,2}}{2} \left(\frac{\partial^2 U^{-1}}{\partial t^2} + 2 \frac{\partial U^{-1}}{\partial t} \frac{\partial U}{\partial t} + U^{-1} \frac{\partial^2 U}{\partial t^2} \right)$$

which leaves

$$\operatorname{Tr} \left(U_{P_{0j}} - 1 \right) + \text{ h.c.} = \operatorname{Tr} \left(U^\dagger_{t+a^t} U_t - 1\!\!1 \right) + \text{ h.c.} = -a^{t\,2} \operatorname{Tr} \left(\frac{\partial U^{-1}}{\partial t} \frac{\partial U}{\partial t} \right).$$

We can improve this by using our expression for zero again, $\frac{\partial U^{-1}}{\partial t} U = -U^{-1} \frac{\partial U}{\partial t}$, so $\frac{\partial U^{-1}}{\partial t} = -U^{-1} \frac{\partial U}{\partial t} U^{-1}$ and

$$\operatorname{Tr} \left(U^\dagger_{t+a^t} U_t - 1\!\!1 \right) + \text{ h.c.} = a^{t\,2} \operatorname{Tr} \left(U^{-1} \dot U U^{-1} \dot U \right),$$

and

$$L = \sum_{\vec n} \frac{-a}{2g^2} Tr \left(U^{-1} \dot U U^{-1} \dot U \right) - V\{U\},$$

where V contains the spacelike placquette terms, without any time derivatives.

To get the Hamiltonian we need first to find the canonical momenta $\Pi = \frac{\delta L}{\delta \dot q}$. In our case the coordinates q are the matrix elements of U on each spacelike link $U_{\ell\, ab}$, where ℓ specifies the link (both spacelike direction and spatial position) and ab are the matrix indices. For convenience we will define the matrix P as the transpose, so

$$(P_\ell)_{cb} := \frac{\delta L}{\delta\left(\dot{U}_\ell\right)_{bc}} = \frac{-a}{g^2}\left(U_\ell^{-1}\dot{U}_\ell U_\ell^{-1}\right)_{cb},$$

$$H = \sum_{\ell,c,b} \dot{U}_{\ell\, bc} P_{\ell\, cb} - L = \sum_\ell \frac{-g^2}{2a} \operatorname{Tr}\left(U_\ell P_\ell U_\ell P_\ell\right) + V(\{U\}).$$

Quantum mechanically, we write the momentum conjugate to q as $-i\frac{\partial}{\partial q}$, so $P_{\ell cb} = -i\frac{\partial}{\partial U_{\ell bc}}$. But P only enters H together with U. So let us define, for each link,

$$(E_\ell)_{ab} = i\left(U_\ell P_\ell\right)_{ab} = U_{\ell\, ac}\frac{\partial}{\partial (U_\ell)_{bc}}.$$

In terms of of the E's, the kinetic energy term in L becomes $\sum_\ell \frac{a}{2g^2}\operatorname{Tr} E_\ell^2$.

The E's are a scaled electric field (ag times the usual field). As a differential operator it is very interesting.

But before we pursue the group theoretic arguments over the interpretation of E, let us meet an objection to the above procedure, that the U_{ab} are not independent real coordinates[2]. Instead, we should write $U = e^{i\omega^i L_i}$ and treat the real, independent ω^i as the coordinates. Then we get

$$\Pi_i = \frac{\delta L}{\delta \dot{\omega}^i} = \frac{\delta \dot{U}_{ab}}{\delta \dot{\omega}^i}\frac{\delta L}{\delta \dot{U}_{ab}} = \frac{\delta U_{ab}}{\delta \omega^i}\frac{\delta L}{\delta \dot{U}_{ab}}$$

where I have made use of $\dot{U}_{ab} = \frac{\delta U_{ab}}{\delta\omega^i}\dot{\omega}^i$.

But $\dfrac{\delta L}{\delta \dot{U}_{ab}} = -\dfrac{a}{g^2}\left(U^{-1}\dot{U}U^{-1}\right)_{ba}$, so

$$\Pi_i = -\frac{a}{g^2}\operatorname{Tr}\left(\frac{\delta U}{\delta\omega^i}U^{-1}\left[\sum_j \dot{\omega}^j\frac{\delta U}{\delta \omega^j}\right]U^{-1}\right)$$

$$\text{and}\quad \dot{\omega}^i\Pi_i = -\frac{a}{g^2}\operatorname{Tr}\left(\dot{\omega}^i\frac{\delta U}{\delta\omega^i}U^{-1}\dot{\omega}^j\frac{\delta U}{\delta\omega^j}U^{-1}\right) = -\frac{a}{g^2}\operatorname{Tr}\left(\dot{U}U^{-1}\dot{U}U^{-1}\right)$$

[2]Of course we have a U and an ω for each link, but we will drop the index ℓ, or $\vec{n},\ell$ that specifies which link, in what follows.

in agreement with what we found by careless manipulation.

One sees, however, that there are only N^2-1 coordinates and momenta for SU(N), not N^2 as on might expect from $P_{\ell\, cb}$. The requirement that U be unitary and of determinant 1 means there are constraints on the $P_{\ell\, cb}$'s — we can vary U only preserving the unitarity and determinant. Interpreting the $P_{\ell\, cb}$ quantum mechanically as $-i\frac{\delta}{\delta U_{\ell\, bc}}$ would be varying into a dimension for which the action is undefined, where $\det U \neq 1$. Our $E_{\ell\, cb}$ do not have that problem. Consider any function f on $n\times n$ unitary matrices u. Each such matrix is $u = e^{i\phi}U$ with $U \in SU(N)$, so we may write $f(u) = \tilde{f}(\phi, U)$ and

$$\left.\frac{\partial \tilde{f}}{\partial U_{ab}}\right|_\phi = \frac{\partial f}{\partial u_{cd}}\left.\frac{\partial u_{cd}}{\partial U_{ab}}\right|_\phi = e^{i\phi}\frac{\partial f}{\partial u_{ab}},$$

so if $E_{ab} = iu_{ac}\dfrac{\partial}{\partial u_{bc}}$, $E_{ab}f = ie^{i\phi}U_{ac}\dfrac{\partial f}{\partial u_{bc}} = iU_{ac}\left.\dfrac{\partial \tilde{f}}{\partial U_{bc}}\right|_\phi$ and has no dependence on how f varies off the $\det u = 1$ subset. So we see that the E_{ab} does not involve a derivative in the direction away from $\det U = 1$, but does form a complete set of N^2-1 first order differential operators on functions of SU(N) matrices.

The N^2-1 momenta conjugate to ω^i are quantum mechanically

$$\Pi_i = -i\frac{\delta}{\delta\omega_i} = -i\frac{\delta U_{ab}}{\delta\omega^i}\frac{\delta}{\delta U_{ab}}.$$

Collectively these clearly span the same space as the E_{ab}. To understand the connection, consider different ways of asking how a function on G varies as we move the element $g\in G$.

We see that Π_i is not the same as E. To understand the connections, consider a function $f: G\to\mathbb{C}$ defined on the group. Then derivatives of f mean finding the change in $f(U)$ under small changes in U. If we write $U = e^{i\omega^i L_i}$, then

$$\frac{\partial}{\partial\omega^i}f(U) = i\Pi_i f.$$

On the other hand, we could consider the change in f as $U \to U'(\omega) = e^{i\omega^i L_i}U$, for infinitesimal ω^i. If we define the differential operators

$$\begin{aligned} E_j f(U) &= -i\left.\frac{\partial}{\partial\omega^j}f\left(e^{i\omega^j L_j}U\right)\right|_{\omega=0} = -i\frac{\partial f(U')}{\partial U'_{ab}}\left.\frac{\partial U'_{ab}}{\partial\omega_j}\right|_{\omega_j=0} = \frac{\partial f(U)}{\partial U_{ab}}i(L_j)_{ac}U_{cb} \\ &= (L_j)_{ac}U_{cb}\frac{\partial}{\partial U_{ab}}f = \mathrm{Tr}(L_j E)f. \end{aligned}$$

So we see that the electric fields $E_{ab} = U_{\ell\,ac} P_{\ell\,cb}$ which we defined earlier are just linear combinations of the left-handed differential operators E_j, with $E = \sum_j E_j L_j$.

For completeness, let us also define a right derivative on the space of functions from $G \to \mathbb{C}$ by

$$\begin{aligned} R_j f(U) &= -i\,\frac{\partial}{\partial\omega^i} f\left(U e^{i\omega_j L_j}\right)\Big|_{\omega=0} = \frac{\partial f}{\partial U_{cb}}\left(-i\frac{\partial}{\partial\omega^i}\left(U e^{i\omega_j L_j}\right)_{cb}\right)\Big|_{\omega=0} \\ &= \frac{\partial f}{\partial U_{cb}} U_{ca}\left(L_j\right)_{ab}. \end{aligned}$$

This is not exactly the same as E_j If we make a matrix from R_j in the same way as we did for E_j,

$$\begin{aligned} R_{ab} &= \sum_j R_j \tau_{j\,ab} = U_{cb}\frac{\delta}{\delta U_{ca}} \\ \text{and recall} \quad E_{ab} &= U_{ac}\frac{\delta}{\delta U_{bc}}, \end{aligned}$$

so

$$Rf = \frac{\delta f}{\delta U^T} U = U^{-1} U \frac{\delta f}{\delta U^T} U = U^{-1}(Ef)U.$$

Note that the E_j, R_j and Π_j are each sets of N^2-1 first order differential operators on the space of functions from $G \to \mathbb{C}$, an N^2-1 dimensional space. Thus they are linear combinations of each other. The connection between them is associated with the adjoint representation of the group.

We will write $\left[\mathcal{S}(L_\ell)\right]_{jk}$ for the adjoint representation of the generators, so $\left[\mathcal{S}(L_\ell)\right]_{jk} = i c_{j\ell}{}^k$. Let us call the adjoint representation of the group S:

$$S_{jk}(e^{i\omega^\ell L_\ell}) = \left[e^{i\omega^\ell \mathcal{S}(L_\ell)}\right]_{jk}.$$

[Note: $\mathcal{S}$ and S were both previously called Γ^{adj}. Also it is hard to distinguish $\mathcal{S}$ from S.] These will come up in the connections we are seeking.

14.1 Relation of R_j to E_j

Now $U^{-1} L_\ell U$ is a linear combination of the L's, so $U^{-1} L_\ell U = M_{\ell m}(U) L_m$. We can show $M_{\ell m} = S_{\ell m}$.

Proof: Let $U(\lambda) = e^{i\lambda\omega^j L_j}$, $M(\lambda) := M(U(\lambda))$. Then $e^{-i\lambda\omega^j L_j} L_\ell e^{i\lambda\omega^k L_k} = M_{\ell m}(\lambda) L_m$. Differentiate with respect to λ:

$$e^{-i\lambda\omega^j L_j} \underbrace{[L_\ell, i\omega^k L_k]}_{-\omega^k c_{\ell k}{}^n L_n} e^{i\lambda\omega^j L_j} = \frac{dM_{\ell m}(\lambda)}{d\lambda} L_m = -\omega^k c_{\ell k}{}^n M_{nm}(\lambda) L_m.$$

The L_m are linearly independent, so

$$\left(\frac{dM(\lambda)}{d\lambda}\right)_{\ell m} = -\omega^k c_{\ell k}{}^n M_{nm}(\lambda) = i\omega^k \left[\mathcal{S}(L_k) M\right]_{\ell m}.$$

This is a differential equation with solution

$$M(\lambda) = e^{i\lambda\omega^k \mathcal{S}(L_k)} K$$

with initial condition $M_{\ell m}(0) = \delta_{\ell m} = 1\!\!\mathrm{I} = K$, so $M(\lambda) = e^{i\lambda\omega^k \mathcal{S}(L_k)} = S(U(\lambda))$. Thus

$$e^{-i\omega^k L_k} L_\ell e^{i\omega^k L_k} = S_{\ell m}(e^{i\omega^b L_b}) L_m = \left(e^{i\omega^b \mathcal{S}(L_b)}\right)_{\ell m} L_m.$$

So $U^{-1} L_\ell U = S_{\ell m}(U) L_m$ and by exponentiation, $U^{-1} e^{i\omega^k L_k} U = e^{i\omega^k S_{km}(U) L_m}$. Now reconsider

$$\begin{aligned} R_\ell f(U) &= -i \frac{\partial}{\partial \omega^\ell} f\left(U e^{i\omega^k L_k}\right)\Big|_{\omega=0} \\ &= -i \frac{\partial}{\partial \omega^\ell} f\left(e^{i\omega^k S_{km}(U^{-1}) L_m} U\right)\Big|_{\omega=0} \\ &= -i \underbrace{\frac{\partial \omega'^n}{\partial \omega^\ell}}_{S_{\ell n}(U^{-1})} \frac{\partial}{\partial \omega'^n} f\left(e^{i\omega'^m L_m} U\right)\Bigg|_{\omega=0} \end{aligned}$$

where $\omega'^m = \omega^k S_{km}(U^{-1})$. So

$$R_\ell f(U) = S_{\ell m}(U^{-1}) E_m f(U), \quad R_\ell = S_{\ell m}(U^{-1}) E_m, \quad \text{and} \quad E_m = S_{m\ell}(U) R_\ell,$$

as $S^{-1}(U^{-1}) = S(U)$.

Why is E_ℓ better than R_ℓ? In the Hamiltonian we have $\sum E_\ell^2$, why not $\sum R_\ell^2$?

$$\begin{aligned}
\sum E_m^2 &= S_{m\ell}(U)R_\ell S_{mk}(U)R_k \\
&= S_{m\ell}(U)S_{mk}(U)R_\ell R_k - iS_{m\ell}\left[\frac{\partial}{\partial\omega^\ell}S_{mk}\left(Ue^{i\omega^j L_j}\right)\Big|_{\omega=0}\right]R_k.
\end{aligned}$$

Note that $\mathcal{S}$ is hermitean and imaginary so $S = e^{i\omega^j\mathcal{S}(L_j)}$ is unitary and real, hence orthogonal, and $S_{m\ell}(U)S_{mk}(U) = \delta_{\ell k}$. Also $S_{mk}(Ue^{i\omega^j L_j}) = S_{mn}(U)S_{nk}(e^{i\omega^j L_j})$, so $-i\,\frac{\partial}{\partial\omega^\ell}S_{mk}(Ue^{i\omega L})\big|_{\omega=0} = S_{mj}(U)\mathcal{S}_{jk}(L_\ell)$, so the second term is $-iS_{m\ell}\,(-S_{mj})\,\mathcal{S}_{jk}(L_\ell)R_k = \delta_{\ell j}(ic_{j\ell}{}^k)R_k$. Thus

$$\sum E_m^2 = \sum R_\ell R_k\delta_{\ell k} + i\delta_{\ell m}c_{m\ell}{}^j R_j = \sum R_j^2,$$

by the antisymmetry of c. So in fact the Hamiltonian does not distinguish between $\sum E_\ell^2$ and $\sum R_\ell^2$, as they are the same thing.

We can also ask what the connection is between E_ℓ and $\frac{\partial}{\partial\omega^\ell}$. To find the connection we need to develop a formula for $\frac{\partial}{\partial x}e^{A(x)}$ for an operator A which need not commute with its derivative. The formula is

$$\frac{\partial}{\partial x}e^{A(x)} = \int_0^1 d\alpha\, e^{\alpha A(x)}\,\frac{\partial A(x)}{\partial x}\,e^{(1-\alpha)A(x)} \tag{14.1}$$

which you proved for homework. Then around a fixed element $U_0 = e^{i\omega_0^\ell L_\ell}$, we may write

$$e^{i\omega^\ell L_\ell} = e^{i\rho^\ell L_\ell}e^{i\omega_0^\ell L_\ell}$$

with $\rho \to 0$ as $\omega \to \omega_0$. Thus

$$\frac{\partial}{\partial\omega^\ell}f\left(e^{i\omega^\ell L_\ell}\right) = \frac{\partial\rho^m}{\partial\omega^\ell}\frac{\partial}{\partial\rho^m}f\left(e^{i\rho^m L_m}U_0\right) = \frac{\partial\rho^m}{\partial\omega^\ell}iE_m f.$$

The partial derivative matrix can be evaluated with this formula as applied to $f(U) = U$,

$$\frac{\partial}{\partial\omega^\ell}e^{i\omega^k L_k} = i\frac{\partial\rho^m}{\partial\omega^\ell}L_m e^{i\omega^k L_k} = \int_0^1 d\alpha\; e^{i\alpha\omega^k L_k}\;iL_\ell\;e^{i(1-\alpha)\omega^k L_k}$$

or

$$\frac{\partial\rho^m}{\partial\omega^\ell}L_m = \int_0^1 d\alpha\; e^{i\alpha\omega^k L_k}L_\ell e^{-i\alpha\omega^k L_k} = \int_0^1 d\alpha\,S_{\ell m}\left(e^{-i\alpha\omega^k L_k}\right)L_m,$$

or

$$\frac{\partial \rho^m}{\partial \omega^\ell} = \int_0^1 d\alpha \left[e^{-i\alpha\omega^k \mathcal{S}(L_k)} \right]_{\ell m} = \left[\frac{1 - e^{-i\omega^k \mathcal{S}(L_k)}}{i\omega^k \mathcal{S}(L_k)} \right]_{\ell m} .$$

so

$$\frac{\partial}{\partial \omega^\ell} = \left[\frac{1 - e^{-i\omega^k \mathcal{S}(L_k)}}{\omega^k \mathcal{S}(L_k)} \right]_{\ell m} E_m$$

gives the connection of the left derivative with the ordinary partial derivative. [Note: the fraction in square brackets is well defined even though $\omega^k \mathcal{S}(L_k)$ is likely not an invertible matrix, so dividing by it is dubious. But the expression in the bracket has a well defined power series expansion and thus a well defined meaning.]

We have given a great deal of discussion to our Hamiltonian operator. We now turn to the states on which this operator acts. A state can be specified by giving
(a) at each site, the occupation number for the fermions at that site
(b) on each link, a group element. We focus on this.

We are not accustomed to having dynamics with canonical coordinates defined on a compact space like $U \in G$, except in advanced courses in classical mechanics (*e.g.* . the orientation of a rigid body). In quantum mechanics we know that the momentum is related to translations, so the ordinary kinetic energy term $\sum_i p_i^2/2m$ is connected with a space of q's which is translation invariant in each p_i. Our kinetic energy is not of that form, as the E_ℓ's don't commute and the $\partial/\partial\omega^\ell$'s do not enter H in this simple fashion. This is consistent with the fact that our space of U's is just not a Euclidean $N^2 - 1$ dimensional space. Now quantum mechanics can be viewed as a functional integral over all possible configurations, which requires a *weight* for the configurations. In a deep sense this weight is connected with the kinetic energy term in the Lagrangian density. We can sidestep this issue (temporarily) by observing that the only sensible way to integrate over the U's is in a group invariant way. In terms of eigenstates $|U_{ab}\rangle$ of U (on a given link), we need to be able to write

$$1\!\!1 = \int d\mu(U) \, |U_{ab}\rangle \, \langle U_{ab}| \, ,$$

or in terms of the parameters ω^ℓ, $d\mu(U) = d\mu(\omega^\ell)$, the measure must be group invariant, $\int d\mu(\omega) f(U(\omega)) = \int d\mu(\omega) f(GU(\omega))$ for any function $f : G \to \mathbb{C}$

and any fixed element G in the group. This is also what we needed long ago to define the orthogonality of group representations.

So now we leave physical applications and turn to the problem of finding the **Haar** or **Hurwitz** measure $d\mu(\omega^\ell)$ satisfying the invariance.

Let us write $d\mu(\omega) = h(\omega)d\omega^1 \cdots d\omega^n$ for an n dimensional Lie group. For a fixed G and ω, the group element $U' = GU(\omega) = U(\omega') = e^{i\omega'^j L_j}$, where the ω'^j are functions of the ω's (and also of G, but we are holding G fixed). If we change variables on the right hand side of our invariance equation, we get

$$\int h(\omega) f\left(U(\omega)\right) d^n\omega = \int h(\omega) f\left(U(\omega')\right) \left|\frac{\partial \omega^j}{\partial \omega'^k}\right| d^n\omega'$$

where $|\partial/\partial|$ is the Jacobian determinant. Invariance for arbitrary functions f clearly requires

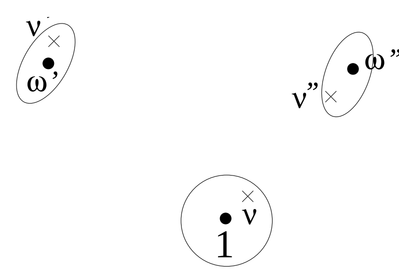
$$h(\omega') = h(\omega) \left|\frac{\partial \omega^j}{\partial {\omega'}^k}\right|.$$

Let $h(0) = c$. Then h at any other point ω is determined by taking $G = e^{i\omega^j L_j}$,

$$h(\omega) = c \left|\frac{\partial {\nu'}^k}{\partial \nu^j}\right|^{-1}_{\nu=0},$$

where $e^{i{\nu'}^k L_k} = e^{i\omega^k L_k} e^{i\nu^k L_k}$.

This is a necessary condition for the invariance. To show it is also sufficient, we need to show that the relation holds for all ω. Consider the measure at two points, ω' and ω''. We need to show that if $U(\omega'') = GU(\omega')$ for a fixed G,



$$h(\omega'') = h(\omega') \left|\frac{\partial \nu'}{\partial \nu''}\right|_{\nu''=\omega''}.$$

Let $G' = e^{i{\omega'}^k L_k}$ and $G'' = e^{i{\omega''}^k L_k}$ and $U(\nu') = G'U(v)$ and $U(\nu'') = G''U(\nu)$. Then $U(\nu'') = G''(G')^{-1}U(\nu') = GU(\nu')$, so $G = G''(G')^{-1}$

$$\left|\frac{\partial \nu''}{\partial \nu'}\right|_{\nu'=\omega'} = \left|\frac{\partial \nu''}{\partial \nu}\right|_0 \Big/ \left|\frac{\partial \nu'}{\partial \nu}\right|_0 = \left(\frac{c}{h(\omega'')}\right) \Big/ \left(\frac{c}{h(\omega')}\right) = \frac{h(\omega')}{h(\omega'')}$$

which verifies the invariance in general

Now we need to evaluate the Jacobian at $\nu = 0$. As $e^{i\nu'^k L_k} = e^{i\omega^k L_k} e^{i\nu^k L_k}$,

$$\begin{aligned}\left.\frac{\partial}{\partial \nu'^k} e^{i\nu'^k L_k}\right|_{\nu=0} &= \frac{\partial \nu^j}{\partial \nu'^k} e^{i\omega^k L_k} \left.\frac{\partial}{\partial \nu^j} e^{i\nu^k L_k}\right|_{\nu=0} = \frac{\partial \nu^j}{\partial \nu'^k} e^{i\omega^k L_k} i L_j \\ &= \left.\int_0^1 d\alpha\, e^{i(1-\alpha)\nu'^k L_k} i L_j\, e^{i\alpha\nu'^k L_k}\right|_{\nu'=\omega}\end{aligned}$$

so

$$\frac{\partial \nu^j}{\partial \nu'^k} L_j = \int_0^1 d\alpha e^{-i\alpha\omega^j L_j} L_k e^{i\alpha\omega^j L_j} = \left[\frac{1 - e^{i\omega^m \mathcal{S}(L_m)}}{-i\omega^m \mathcal{S}(L_m)}\right]_{kj} L_j.$$

Thus

$$h(\omega) = c \det \frac{\partial \nu}{\partial \nu'} = c \det \left(\frac{e^{i\omega^m \mathcal{S}(L_m)} - 1}{i\omega^m \mathcal{S}(L_m)}\right).$$

For any fixed (real) ω, $\omega^m \mathcal{S}(L_m)$ is a hermitean matrix and can be diagonalized. If λ_j are the real eigenvalues, we have

$$h(\omega) = c \prod_j \frac{e^{i\lambda_j} - 1}{i\lambda_j} = \prod_j \frac{\sin(\lambda_j/2)}{\lambda_j/2} \prod_j e^{i\lambda_j/2}.$$

But $\omega^m \mathcal{S}(L_m)$ is also imaginary, while still hermitean, so it is antisymmetric and has zero trace. This is preserved by similarity, so $\sum_j \lambda_j = 0$, and

$$h(\omega) = c \prod_j \frac{\sin(\lambda_j/2)}{\lambda_j/2}.$$

Let us consider an explicit example, SU(2). Any $U = e^{i\vec{\omega}\cdot\vec{\tau}/2}$ is equivalent (by similarity) to one along the z axis, with the same $\vec{\omega}^{\,2}$. Then $\omega^j \mathcal{S}(L_j) = |\omega| \mathcal{S}(L_3)$. $\mathcal{S}$ is the adjoint representation, so L_3 has the eigenvalues $\pm 1, 0$. The zero eigenvalue gives 1 in the product (recall the note on the definition of the [] expression) and the others each give $2\sin(|\omega|/2)/|\omega|$, so all together

$$h(\omega) = 4c \frac{\sin^2(|\omega|/2)}{\omega^2}.$$

Choosing $c = 1$, we may take the invariant volume to be

$$\int_{|\omega| \leq 2\pi} d\omega^1 d\omega^2 d\omega^3 \frac{4\sin^2(|\omega|/2)}{|\omega|^2}.$$

If we used spherical coordinates for $\vec{\omega}$, this gives

$$\int_0^{2\pi} d\omega\, d^2\Omega\ 4\sin^2(|\omega|/2),$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ as usual. We note that the volume with $|\omega| \geq 2\pi - \epsilon$ is $\approx \frac{4\pi}{3}\epsilon^3$, the same as the volume with $|\omega| \leq \epsilon$, and not ϵ times the ordinary area of the sphere, that is, $4\pi(2\pi)^2\epsilon$. The metric says points with $\omega \approx 2\pi$, even in very different directions, are close together, because for $\omega = 2\pi$, all these ω's correspond to the same $U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Note $h(\omega)$ is exactly the metric[3] on S^3 induced by embedding in E^4, with the angle with respect to the 4'th axis $\omega/2$.

Some comments:

(a) For SU(3), not all directions are equivalent. So there is no equivalent of $|\omega|$.

(b) For many applications, we needed the existence of the invariant measure but not the actual expression for it. For example, in our orthogonality proofs.

(c) Most discussions of SU(2) parameterize the group in terms of Euler angles rather than in the way we did, the angle of the rotation and the direction of the axis.

(d) Note that the factor in h is just the determinant of the matrix which connects the left derivatives to the ordinary derivatives. This is not an accident, but rather has to do with this relation defining a metric (not just a measure) on the group space, and the measure given by the determinant of the metric.

[3] See "hypersph" in Supplementary Notes.