Chapter 8

Dynkin Diagrams

We now describe how to use the constraints

$$\frac{\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} = \frac{q-p}{2}$$

where p is the number of times $E_{\vec{\alpha}}$ can hit $\left|\vec{\beta}\right\rangle$ without annihilating, and q is the number of times $E_{-\vec{\alpha}}$ can. This is encoded in the **Cartan matrix** for the simple roots,

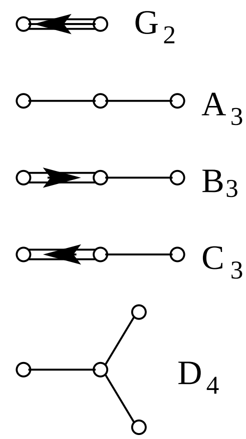
$$A_{ji} = \frac{2\vec{\alpha}_j \cdot \vec{\alpha}_i}{\alpha_i^2} = q - p.$$

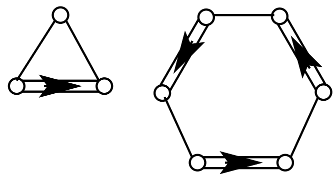
We will draw a Dynkin diagram for each semisimple compact finite-dimensional Lie group in which each simple root will be represented by a dot or small circle, and they will be connected by bonds according to the p values. As we saw, these p values are quite limited, but we will also find constraints on the way even these limited number of bonds can be put together. We shall see that many conceivable diagrams are forbidden, and therefore that the set of all such groups is quite limited.

For each simple root draw a small open circle.

If α and β are two simple roots, and $-2\dfrac{\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} = 2$ or 3, draw 2 or 3 lines, respectively, between their corresponding circles, with an arrow from α to β. That is, from the shorter root to the longer. If $-2\dfrac{\vec{\alpha} \cdot \vec{\beta}}{\alpha^2} = -2\dfrac{\vec{\alpha} \cdot \vec{\beta}}{\beta^2} = 1$, draw a single line without an arrow between the two roots. If $\vec{\alpha} \cdot \vec{\beta} = 0$, do not connect the two circles corresponding to α and β.

With these rules we can draw diagrams such as these, and investigate if there is a group corresponding to that diagram. We will see, however, that there are restrictions on what is possible. It will turn out that the diagrams on the right do correspond to Lie groups, while those below do not.



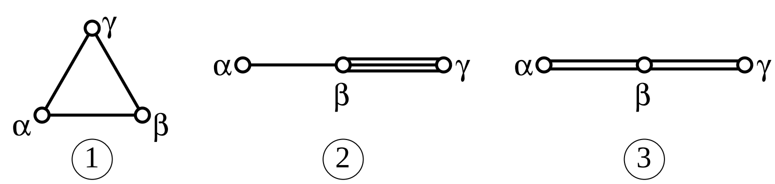


8.1 Classification of Simple Lie Algebras

We have seen that given any algebra, there exist m simple roots, where m is the **rank** of the algebra, with relations that can be represented by a Dynkin diagram.

Let us first show any algebra with a disconnected diagrams cannot be simple. Suppose that the simple roots can be divided into two sets, A and B, with the vertices of A disconnected from B. Then, for $\alpha \in A$ and $\beta \in B$, $E_{-\alpha} \left| E_\beta \right\rangle = 0$, which is always true for simple roots, so $q = 0$. As there is no connection of α and β, $\vec{\alpha} \cdot \vec{\beta} = 0$, and $p = 0$, so $E_\alpha \left| E_\beta \right\rangle = 0$. Each simple root in A commutes with each in B and with its conjugate. The other roots in A are commutators of simple ones, and similarly for B, so all roots in A commute with all roots in B. The group is therefore the direct product of that generated by A and that generated by B. Each of these is a nontrivial invariant subalgebra, so the whole group is not simple. Of course the direct product of simple groups is semisimple, but not simple.

Next, we rule out diagrams with three simple roots connected as shown, whether or not there are additional connections to other roots.



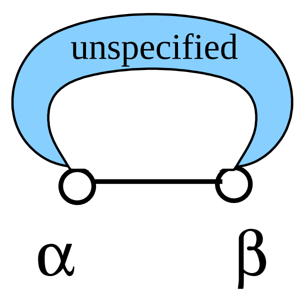
The angles between these roots are as given in the table, with the angles even bigger if any additional lines are added. Each is a vector with its tail at the origin. As the sum of three angles between three such vectors must be $\leq 360°$, and equal to $360°$ only if the three vectors lie in a plane, we see the three roots must lie in a plane, and therefore cannot be linearly independent, which is impossible.

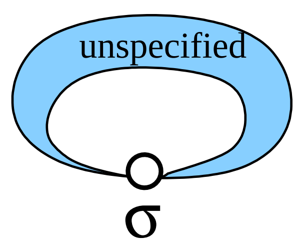
	(1)	(2)	(3)
$\theta_{\alpha\beta}$	120°	120°	135°
$\theta_{\beta\gamma}$	120°	150°	135°
$\theta_{\gamma\alpha}$	120°	90°	90°
$\sum \theta$	360°	360°	360°

Thus the only diagram with a "triple bond" is the diagram for G_2.

A set of vectors which corresponds to the Dynkin diagram rules, having the magnitudes and angles as indicated by the diagram, is called a Π system, without worrying about whether each Π system corresponds to an algebra.

Now consider a Π system containing two simple roots α and β connected to each other with a single line, with unspecified connections to the rest of the system. Then the Π system which results from contracting the line, collapsing α and β to a single root $\vec{\sigma} = \vec{\alpha} + \vec{\beta}$. is also a Π system. For $\sigma^2 = \alpha^2 = \beta^2$ because they were connected by a single line, so are of equal magnitude with an angle of $120°$ between them. For each γ in the unknown, γ cannot be connected to both α and β, because diagram (1) above is excluded, and is not part of any Π system. If γ was disconnected from β, $\vec{\gamma} \cdot \vec{\beta} = 0$, so



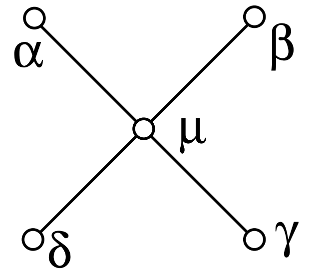


$$\vec{\gamma} \cdot \vec{\sigma} = \vec{\gamma} \cdot (\vec{\alpha} + \vec{\beta}) = \vec{\gamma} \cdot \vec{\alpha},$$

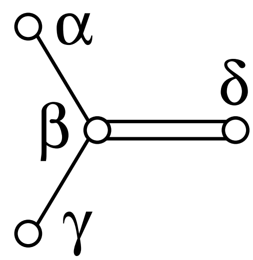
so γ is connected to σ with the same connection that it had been to α.

Corrollaries:

- No Π system contains more than 1 double bond (or else collapsing whatever connected them would produce diagram (3) above).
- No Π system contains contains a closed loop, for if so it could be collapsed to diagram (1).



No Π system contains a root linked to four others with single links. For then
$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = \mu^2$,
and $\vec\mu \cdot \vec\alpha = -\mu^2/2 = \vec\mu \cdot \vec\beta = \vec\mu \cdot \vec\gamma = \vec\mu \cdot \vec\delta$,
$\vec\alpha \cdot \vec\beta = \vec\alpha \cdot \vec\gamma = \vec\alpha \cdot \vec\delta = \vec\beta \cdot \vec\gamma = \vec\beta \cdot \vec\delta = \vec\gamma \cdot \vec\delta = 0$,
so $(\vec\alpha + \vec\beta + \vec\gamma + \vec\delta + 2\vec\mu)^2 = \mu^2(4 + 4 - 8) = 0$, and the five roots are not all independent.



No Π system contains a root connected to three others, including one double bond, regardless of direction. For $\alpha^2 = \beta^2 = \gamma^2$, $\vec\alpha \cdot \vec\beta = \vec\beta \cdot \vec\gamma = -\frac{1}{2}\beta^2$, $\vec\alpha \cdot \vec\gamma = \vec\alpha \cdot \vec\delta = \vec\gamma \cdot \vec\delta = 0$. The angle between $\vec\beta$ and $\vec\delta$ is $135°$. Let $\hat\delta$ be in the direction of $\vec\delta$ but with the length of $\vec\beta$ (changing it by $\sqrt{2}$ or its inverse). Then $\vec\beta \cdot \hat\delta = \beta^2 \cos 135° = -\beta^2/\sqrt{2}$, and

$$\begin{aligned}(\vec\alpha + \vec\gamma + 2\vec\beta + \sqrt{2}\hat\delta)^2 &= \alpha^2 + \gamma^2 + 4\beta^2 + 2(\hat\delta)^2 + 4\vec\alpha \cdot \vec\beta + 4\vec\gamma \cdot \vec\beta + 4\sqrt{2}\vec\beta \cdot \hat\delta \\ &= \beta^2(1 + 1 + 4 + 2 - 2 - 2 - 4) = 0,\end{aligned}$$

so again the roots are not linearly independent, which is impossible [1].

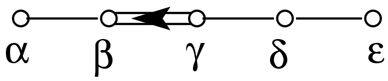
Corrolaries:

- No root is linked directly to four others. We have already shown that for four single bonds, but if one is double, contracting one of the others reduces to the case just excluded.
- No Π system contains both a branch point and a double bond, because contracting what is between them gives what we just excluded, and no Π system contains more than one branch point, because contracting what connects them leaves a root linked to four others.

So thus far we see that all Π systems are either
(a) G_2
(b) a linear chain with at most one double bond
(c) a chain with one branch point and only single bonds.

[1] How does one find combinations like that to prove dependence? If we suspect that $\vec v_i$ might be linearly dependent, so $\exists k_i \ni \sum k_i \vec v_i = 0$, and if we know all the dot products, $\vec v_i \cdot \vec v_j$, we can ask for the minimum of the square $\left(\sum k_i \vec v_i\right)^2 \geq 0$. As that is a quadratic expression in the k_i, the equations for minimizing it are linear homogeneous equations in the k_i, and finding a nonzero solution provides a test for linear dependence of the $\vec v_i$.

Now we will limit choices (b) and (c). First consider (b).

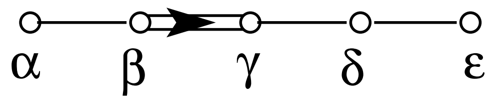
No diagram contains 

$\alpha^2 = \beta^2 = 2\gamma^2 = 2\delta^2 = 2\epsilon^2$, $\vec\alpha\cdot\vec\beta = -\frac{1}{2}\beta^2 = -\gamma^2$, $\vec\beta\cdot\vec\gamma = -\gamma^2$, $\vec\gamma\cdot\vec\delta = \vec\delta\cdot\vec\epsilon = -\frac{1}{2}\gamma^2$, the other dot products vanish, and

$$(\vec\alpha + 2\vec\beta + 3\vec\gamma + 2\vec\delta + \vec\epsilon)^2 = \gamma^2(2+8+9+4+1-4-12-6-2) = 0,$$

and again the five roots are not linearly independent.

If we reverse the arrow $2\alpha^2 = 2\beta^2 = \gamma^2 = \delta^2 = \epsilon^2$, $\vec\alpha\cdot\vec\beta = -\frac{1}{2}\alpha^2$, $\vec\beta\cdot\vec\gamma = -\beta^2$, $\vec\gamma\cdot\vec\delta = \vec\delta\cdot\vec\epsilon = \frac{1}{2}\gamma^2 = -\alpha^2$



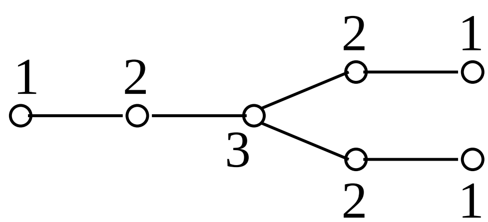
$$(2\vec\alpha + 4\vec\beta + 3\vec\gamma + 2\vec\delta + \vec\epsilon)^2 = \alpha^2(4+16+18+8+2-8-24-12-4) = 0,$$

so once again the five roots are not linearly independent, and this is forbidden.

Thus the only double bond appears at the end of the chain or else we have only the group F_4: 

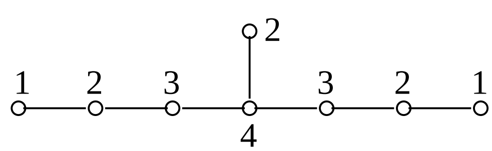
Now consider branches. Here all the simple roots have the same magnitudes, say 1, and if connected have a dot product of $-\frac{1}{2}$.

If we add the roots α_i with weights w_i and square, $\left(\sum w_i\alpha_i\right)^2 = \sum w_i^2 - \sum_{\rm nn} w_i w_j$, where $\sum_{\rm nn}$ means sum over nearest neighbors (each pair once). First consider seven roots connected as shown, with weights w_i as indicated.



$$\left(\sum w_i\alpha_i\right)^2 = 1+4+9+4+1+4+1-(2+6+6+6+2+2) = 0.$$

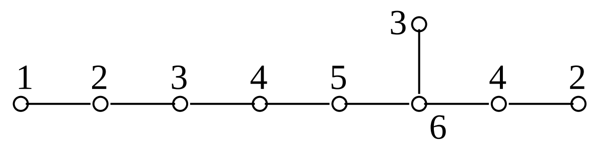
So again this cannot be part of the Dynkin diagram of a group, and the shortest branch can have only one attached root. Now consider whether both of the other branches can have three or more.



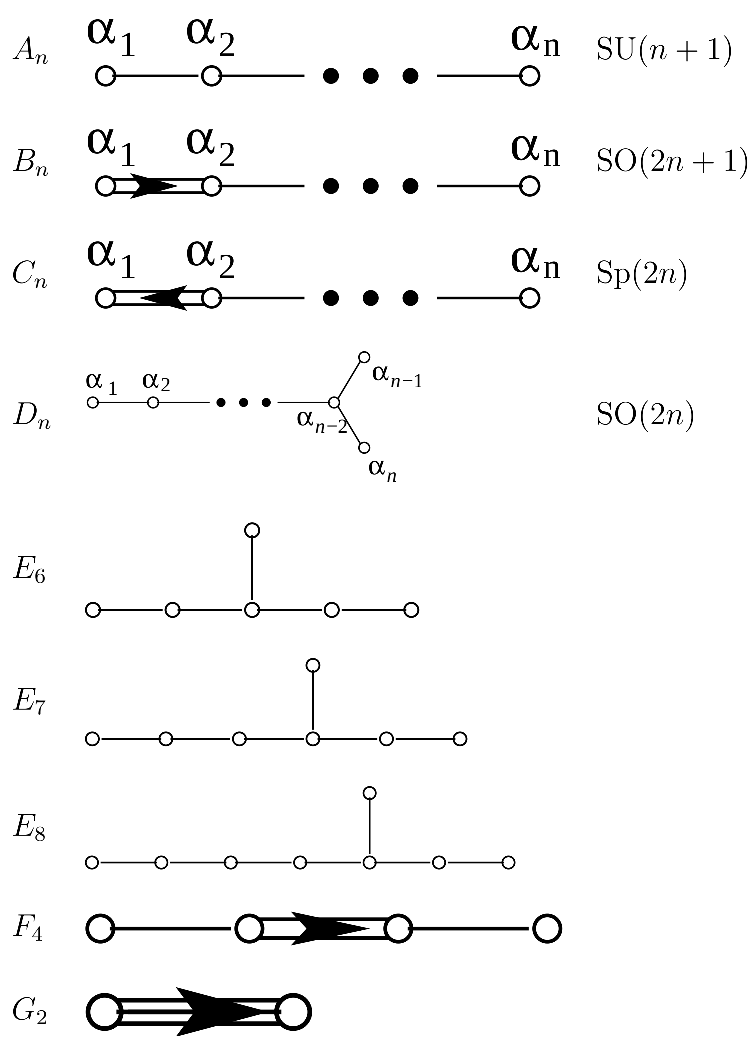
$$\left(\sum w_i\alpha_i\right)^2 = (1{+}4{+}9{+}16{+}4{+}9{+}4{+}1) - (2{+}6{+}12{+}8{+}12{+}6{+}2) = 48-48 = 0,$$

so the next shortest branch has at most 2 attached roots.

Next, if the two shortest are as long as possible, is there a limit to the third branch? $(1{+}4{+}9{+}16{+}25{+}36{+}9{+}16{+}4) - (2{+}6{+}12{+}20{+}30{+}18{+}24{+}8) = 120 - 120 = 0$.



So the longest branch has at most 4 roots attached, if the second longest has two. Thus we have the complete list of simple finite dimensional compact Lie groups



All of these Π systems are allowed and do correspond to a simple Lie algebra.