Physics 618  
Homework #6
Due: Thursday, March 3, 2016 at 4:00 PM

Reminder: There will be a midterm exam on Tuesday, March 8. You will be allowed to use your notes, including my notes, and at most two books.

1  [10 pts] Consider a semisimple Lie algebra \( L \) of rank 2, with simple roots \( \vec{\alpha} \) and \( \vec{\beta} \). Let \( p = -2\vec{\alpha} \cdot \vec{\beta}/\alpha^2 \), and suppose \( |\vec{\alpha}| \leq |\vec{\beta}| \).

a) Show that if \( p = 0 \), the algebra is a direct sum of two algebras which are each an \( SU(2) \). Thus the group is \( SU(2) \times SU(2) \).

b) For \( p = 1 \), find which positive linear combinations of the simple roots are roots. Give a diagram of the root vectors. Check to make sure you have included all roots.

c) Do the same thing for \( p = 2 \).

d) For case (b), with \( p = 1 \), work out the entire algebra, including normalizations. Choose your \( H_1 \) and \( H_2 \) so that one root points along the \( H_1 \) axis.

2  [5 pts] [Note: this is Georgi’s problem 5.B. Also note that \([A, B]_+ \) is the anticommutator, \([A, B]_- = [A, B] \) is the commutator.]

Suppose \( X_a \) are \( N \times N \) matrices satisfying

\[ [X_a, X_b] = i f_{abc} X_c \]

and \( b_i^\dagger \) and \( b_i \) are creation and annihilation operators satisfying (\( i = 1 \) to \( N \))

\[ [b_i, b_j]^\dagger = \delta_{ij} \]
\[ [b_i^\dagger, b_j^\dagger] = [b_i, b_j]_\pm = 0. \]

[That is, these relations hold either for commutators or for anticommutators, but not both simultaneously. Show the following holds in either case.]

Show that the operators

\[ \chi_a = \sum_{i,j} b_i^\dagger [X_a]_{ij} b_j \]

satisfy

\[ [\chi_a, \chi_b] = if_{abc} \chi_c \]

3  [10 pts] [Note: this is Georgi’s problem 6.C] Consider the simple Lie algebra formed by the 10 matrices \( \sigma_a, \sigma_a \tau_1, \sigma_a \tau_3, \) and \( \tau_2 \) where \( \sigma_a \) and \( \tau_a \) are Pauli matrices in orthogonal spaces. That is, we are considering a subspace of the direct product space \( GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \) where the \( \sigma \)'s act on the first space and the \( \tau \)'s on the second. You might want to look at Georgi problem 3.E. Take \( H_1 = \sigma_3 \) and \( H_2 = \sigma_3 \tau_3 \) as the Cartan subalgebra. Find

(a) the roots of the adjoint representation and

(b) the weights of the four dimensional representation generated by these matrices.