1 [10 pts] Consider a semisimple Lie algebra $\mathcal{L}$ of rank 2, with simple roots $\vec{\alpha}$ and $\vec{\beta}$. Let $p = -2\vec{\alpha} \cdot \vec{\beta}/\alpha^2$, and suppose $|\vec{\alpha}| \leq |\vec{\beta}|$.

a) Show that if $p = 0$, the algebra is a direct sum of two algebras which are each an $SU(2)$. Thus the group is $SU(2) \times SU(2)$.

b) For $p = 1$, find which positive linear combinations of the simple roots are roots. Give a diagram of the root vectors. Check to make sure you have included all roots.

c) Do the same thing for $p = 2$.

d) For case (b), with $p = 1$, work out the entire algebra, including normalizations. Choose your $H_1$ and $H_2$ so that one root points along the $H_1$ axis.

2 [5 pts] [Note: this is Georgi’s problem 5.B. Also note that $[A, B]_+$ is the anticommutator, $[A, B]_- = [A, B]$ is the commutator.]

Suppose $X_a$ are $N \times N$ matrices satisfying

$$[X_a, X_b] = i f_{abc} X_c$$

and $b_i^\dagger$ and $b_i$ are creation and annihilation operators satisfying ($i = 1$ to $N$)

$$[b_i, b_j^\dagger]_\pm = \delta_{ij}$$
$$[b_i^\dagger, b_j^\dagger]_\pm = [b_i, b_j]_\pm = 0.$$  

[That is, these relations hold either for commutators or for anticommutators, but not both simultaneously. Show the following holds in either case.]
Show that the operators
\[ \chi_a = \sum_{i,j} b_i^\dagger [X_a]_{ij} b_j \]
satisfy
\[ [\chi_a, \chi_b] = i f_{abc} \chi_c \]

3 [10 pts] [Note: this is Georgi’s problem 6.C] Consider the simple Lie algebra formed by the 10 matrices \( \sigma_a, \sigma_a \tau_1, \sigma_a \tau_3, \) and \( \tau_2 \) where \( \sigma_a \) and \( \tau_a \) are Pauli matrices in orthogonal spaces. That is, we are considering a subspace of the direct product space \( \text{GL}(2, \mathbb{C}) \times \text{GL}(2, \mathbb{C}) \) where the \( \sigma \)'s act on the first space and the \( \tau \)'s on the second. You might want to look at Georgi problem 3.E. Take \( H_1 = \sigma_3 \) and \( H_2 = \sigma_3 \tau_3 \) as the Cartan subalgebra. Find

(a) the roots of the adjoint representation and

(b) the weights of the four dimensional representation generated by these matrices.