Physics 618
Homework #5
Due: Thursday, Feb. 25, 2016 at 4:00 PM

1  [10 pts]  $SU(1,1)$ is the set of $2 \times 2$ complex matrices of determinant 1 which leave invariant the form $|z_1|^2 - |z_2|^2$.
   a) Find all elements in an infinitesimal neighborhood of $\mathbb{I}$, and parameterize them.
   b) What is the dimension of this group? Give a well-chosen basis for the Lie algebra.
   c) Find the structure constants and the Killing form. Is the group semisimple?
   d) Evaluate $e^{\sum \nu_j L_j}$ for finite $\nu_j$, at least well enough to discuss the range of the parameters. Is the group compact? Is it simply connected?

2  [5 pts]  Show that there does not exist a two-dimensional semisimple Lie algebra.

3  [10 pts]  If $R(\vec{\omega})$ is a rotation $\vec{r} \rightarrow \vec{r}' = \vec{r} - \vec{\omega} \times \vec{r}$, which acts on the space of functions on $\mathbb{R}^3$ by $R f = f'$, with $f' (\vec{r}') = f(\vec{r})$, and if for infinitesimal $\vec{\omega}$ we write $R = 1 + i \vec{\omega} \cdot \vec{L}$, a) show that $\vec{L} = -i \vec{r} \times \vec{\nabla}$.
   b) Express this in component form, using $\epsilon_{ijk}$, and verify that the operators $L_i$ satisfy the $SU(2)$ Lie algebra commutation relations.
   c) Show that, in spherical coordinates,

   \[ L_z = -i \frac{\partial}{\partial \phi}, \quad L_\pm = \frac{1}{\sqrt{2}} e^{\pm i\phi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right). \]

   d) Show that if a set of functions $Y_{\ell m}$ transform as $|\ell m\rangle$, then $Y_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta) e^{im\phi}$. Find an equation to give $Y_{\ell'-\ell}(\theta)$ up to a multiplicative constant, and a recursion relation to determine the other $Y_{\ell m}$.

4  [5 pts]  Complete the calculation begun in class for the Clebsch-Gordon coefficients $\left(1 \frac{1}{2} \ell m |1 \frac{1}{2} m_1 m_2\right)$.