Physics 616: Homework 7 Solution

P&S problems 11.2, 12.1, 12.2

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April 23, 2008

11.2 Zeroth-order natural relation

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}(\phi^2)^2 + \bar\psi(i\partial\!\!\!/)\psi - g\bar\psi(\phi^1 + i\gamma^5\phi^2)\psi \tag{1}$$

(a) $U(1)$ symmetry

Defining

$$\tilde\phi = \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) \qquad \psi_R = \frac{1}{2}(1+\gamma^5)\psi \qquad \psi_L = \frac{1}{2}(1-\gamma^5)\psi \tag{2}$$

$$\mathcal{L} = \partial_\mu\tilde\phi^*\partial^\mu\tilde\phi + \mu^2\tilde\phi^*\tilde\phi - \lambda(\tilde\phi^*\tilde\phi)^2 + \bar\psi(i\partial\!\!\!/)\psi - g\sqrt{2}\left(\bar\psi_L\tilde\phi\psi_R + \bar\psi_R\tilde\phi^*\psi_L\right) \tag{3}$$

which is manifestly invariant under

$$\tilde\phi \to e^{i\alpha}\tilde\phi \qquad \psi_R \to e^{-i\alpha/2}\psi_R \qquad \psi_L \to e^{i\alpha/2}\psi_L \tag{4}$$

(b) $\langle\,\phi^1(x)\,\rangle = v$

$$\phi^1(x) = v + \sigma(x) \qquad \phi^2(x) = \pi(x) \qquad \mu^2 = \lambda v^2 \tag{5}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}(2\lambda v^2)\sigma^2 + \bar\psi(i\partial\!\!\!/ - gv)\psi - g\bar\psi\sigma\psi - g\bar\psi i\gamma^5\pi\psi \tag{6}$$

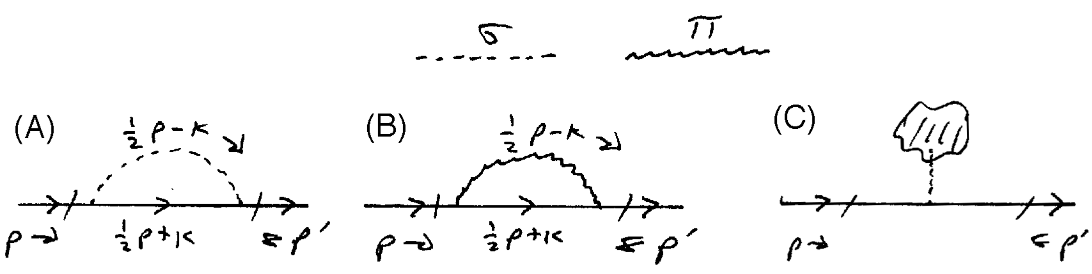
$$- \lambda v(\sigma^3 + \sigma\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \tag{7}$$

$$m_f = gv \qquad m_\sigma^2 = 2\lambda v^2 \tag{8}$$

$$\mathcal{L}_E = -\frac{1}{2}\partial_\mu\pi\partial^\mu\pi - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}(2\lambda v^2)\sigma^2 + \bar\psi(-i\partial\!\!\!/ + gv)\psi + g\bar\psi\sigma\psi + g\bar\psi i\gamma^5\pi\psi \tag{9}$$

$$+ \lambda v(\sigma^3 + \sigma\pi^2) + \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \tag{10}$$

(c) one-loop radiative correction to m_f



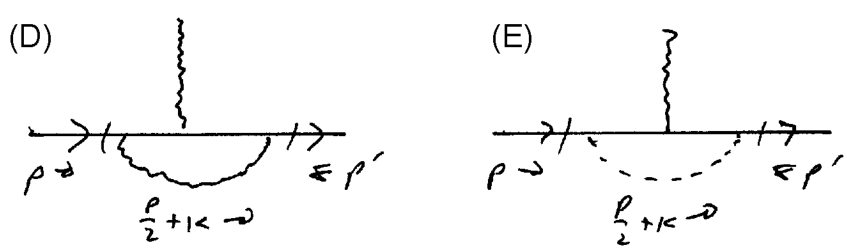
$$
\begin{aligned}
(A) &= (-1)(-g)^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-(\frac{1}{2}p-k)^2+m_\sigma^2} \frac{1}{\frac{1}{2}\not{p}+\not{k}+m_f} && (11)\\
&= -g^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-(\frac{1}{2}p-k)^2+m_\sigma^2} \frac{-\frac{1}{2}\not{p}-\not{k}+m_f}{-(\frac{1}{2}p+k)^2+m_f^2} && (12)\\
&= g^2 \left(\frac{1}{2}\not{p}-m_f\right) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2} + \mathit{finite} && (13)
\end{aligned}
$$

$$
\begin{aligned}
(B) &= (-1)(-ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-(\frac{1}{2}p-k)^2} \gamma^5 \frac{1}{\frac{1}{2}\not{p}+\not{k}+m_f} \gamma^5 && (14)\\
&= g^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-(\frac{1}{2}p-k)^2} \frac{\frac{1}{2}\not{p}+\not{k}+m_f}{-(\frac{1}{2}p+k)^2+m_f^2} && (15)\\
&= g^2 \left(\frac{1}{2}\not{p}+m_f\right) \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2} + \mathit{finite} && (16)\\
(C) &= (-1)(-g)\langle\,\sigma(x)\,\rangle && (17)
\end{aligned}
$$

$$
\not{p}+m_f+(A)+(B)+(C) \;=\; Z_\psi^{-1}\not{p}+m_f+g\langle\,\sigma(x)\,\rangle+\mathit{finite} \qquad (18)
$$

$$
m_f^{phys} = Z_\psi\left(m_f+g\langle\,\sigma\,\rangle+\mathit{finite}\right) \qquad (19)
$$

one-loop radiative correction to $g\bar{\psi}\pi\psi$



$$
\begin{aligned}
(D) &= (-1)\int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2}(-g)i\gamma^5 \frac{1}{-\not{k}}(-g)i\gamma^5 \frac{1}{-\not{k}}(-g)i\gamma^5 + \mathit{finite} && (20)\\
&= -ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2} \frac{1}{-\not{k}} \frac{1}{\not{k}} + \mathit{finite} && (21)\\
&= -ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2} + \mathit{finite} && (22)\\
(E) &= (-1)\int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2}(-g)\frac{1}{-\not{k}}(-g)i\gamma^5 \frac{1}{-\not{k}}(-g) + \mathit{finite} && (23)\\
&= ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2} \frac{1}{-\not{k}} \frac{1}{\not{k}} + \mathit{finite} && (24)\\
&= ig^3\gamma^5 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2} + \mathit{finite} && (25)
\end{aligned}
$$

$$
\begin{aligned}
ig\gamma^5 + (D) + (E) &= i\gamma^5(g + \mathit{finite}) && (26)\\
Z_\psi^{-1} Z_\pi^{-1/2} g^{phys} &= g + \mathit{finite} && (27)
\end{aligned}
$$

radiative corrections to $\langle\, \phi^1(x) \,\rangle$

$$
Z_\phi^{1/2} \langle\, \phi^1(x) \,\rangle = Z_\phi^{1/2} v^{phys} = v + \langle\, \sigma(x) \,\rangle \tag{28}
$$

Put the pieces together:

$$
\begin{aligned}
g^{phys} v^{phys} &= Z_\psi Z_\pi^{1/2} (g + \mathit{finite}) Z_\phi^{-1/2} (v + \langle\, \sigma(x) \,\rangle) && (29)\\
&= Z_\psi (m_f + g \langle\, \sigma(x) \,\rangle) + \mathit{finite} && (30)\\
&= m_f^{phys} + \mathit{finite} && (31)
\end{aligned}
$$

I'm being a bit lazy in going from the 1st line to the 2nd line, using my knowledge that the renormalization is symmetric, so the divergent wave-function renormalizations are the same:

$$
\frac{Z_\pi}{Z_\phi} = 1 + \mathit{finite}\,. \tag{32}
$$

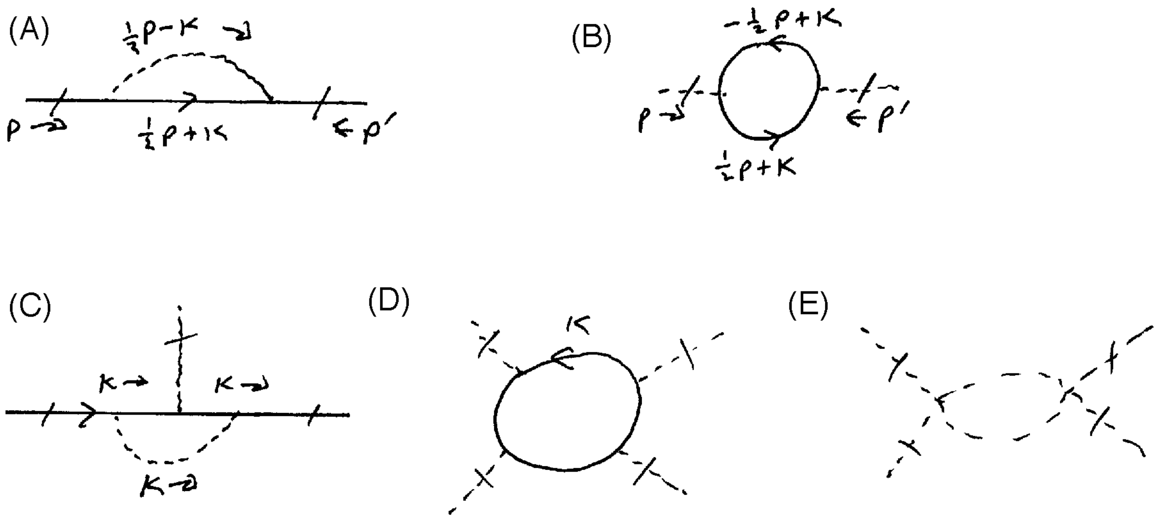
I should really calculate the wave-function renormalizations.

12.1 Beta functions in Yukawa theory

The euclidean action is

$$S_E = \int d^d x_E \left[-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{4!}\phi^4 + \bar{\psi}(-i\partial\!\!\!/)\psi + ig\bar{\psi}\gamma^5\phi\psi \right] \tag{33}$$

There are five 1-loop diagrams:



Diagrams (A) and (B) contribute to the wave-function renormalizations:

$$\begin{aligned}
(A) &= (-1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{-(\frac{1}{2}p-k)^2}(-ig\gamma^5)\frac{1}{\frac{1}{2}p\!\!\!/ + k\!\!\!/}(-ig\gamma^5) && (34)\\
&= g^2\int \frac{d^dk}{(2\pi)^d}\frac{1}{-(\frac{1}{2}p-k)^2}\frac{\frac{1}{2}p\!\!\!/ k\!\!\!/}{-(\frac{1}{2}p+k)^2} && (35)\\
&= \frac{1}{2}g^2 p\!\!\!/ \int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2} && (36)
\end{aligned}$$

$$\begin{aligned}
(B) &= (-1)(-ig)^2(-1)\int \frac{d^dk}{(2\pi)^d}\mathrm{tr}\left[\gamma^5\frac{1}{k\!\!\!/ + \frac{1}{2}p\!\!\!/}\gamma^5\frac{1}{k\!\!\!/ - \frac{1}{2}p\!\!\!/}\right] && (37)\\
&= -g^2\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k+\frac{1}{2}p)^2(k-\frac{1}{2}p)^2}\mathrm{tr}\left[(-1)(k\!\!\!/ + \frac{1}{2}p\!\!\!/)(k\!\!\!/ - \frac{1}{2}p\!\!\!/)\right] && (38)\\
&= -g^2\int \frac{d^dk}{(2\pi)^d}\frac{(-k^2+\frac{1}{4}p^2)2^{d/2}}{(k^2+\frac{1}{4}p^2)^2-(kp)^2} && (39)\\
&= -g^2 2^{d/2}\int \frac{d^dk}{(2\pi)^d}\left[\frac{\frac{1}{4}p^2}{k^4}+\frac{(-k^2)}{k^4}\left(1+\frac{-\frac{1}{2}k^2p^2+(kp)^2}{k^4}\right)\right] && (40)\\
&= -g^2 2^{d/2}\int \frac{d^dk}{(2\pi)^d}\left[\frac{\frac{1}{4}p^2}{k^4}+\frac{1}{(-k^2)}+\frac{(-k^2)}{k^4}\left(\frac{-\frac{1}{2}k^2p^2+\frac{1}{d}k^2p^2}{k^4}\right)\right] && (41)\\
&= -p^2g^2 2^{d/2}\left[\frac{1}{2}+\frac{1}{4}-\frac{1}{d}\right]\int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2} && (42)
\end{aligned}$$

$$
\begin{aligned}
(C) &= (-1)\int \frac{d^dk}{(2\pi)^d}\frac{1}{-k^2}(-ig\gamma^5)\frac{1}{\not k}(-ig\gamma^5)\frac{1}{\not k}(-ig\gamma^5) && (43)\\
&= -ig^3\gamma^5\int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2} && (44)
\end{aligned}
$$

$$
\begin{aligned}
(D) &= (-1)6(-1)\int \frac{d^dk}{(2\pi)^d}\,\mathrm{tr}\left[(-ig\gamma^5)\frac{1}{\not k}(-ig\gamma^5)\frac{1}{\not k}(-ig\gamma^5)\frac{1}{\not k}(-ig\gamma^5)\frac{1}{\not k}\right] && (45)\\
&= 6g^4 2^{d/2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2} && (46)
\end{aligned}
$$

$$
(E) = (-1)\frac{3}{2}(-\lambda)^2\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(-k^2)^2} \qquad (47)
$$

$$
I_1 = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(-k^2)^2} = \frac{1}{(4\pi)^{d/2}}\ln\left(\frac{\Lambda_0}{\Lambda}\right) = \frac{1}{(4\pi)^{d/2}}\frac{\Lambda^{d-4}}{4-d} \qquad (48)
$$

$$
\begin{aligned}
(A) + \not p\,\delta Z_\psi &= \textit{finite} && (49)\\
(B) + (-p^2)\delta Z_\phi &= \textit{finite} && (50)\\
(C) + i\gamma^5\delta g &= \textit{finite} && (51)\\
(D) + (E) + \delta\lambda &= \textit{finite} && (52)
\end{aligned}
$$

$$
\begin{aligned}
Z_\psi &= 1 - g^2 I_1 && (53)\\
Z_\phi &= 1 - 4g^2 I_1 && (54)\\
\delta g &= 2g^3 I_1 && (55)\\
\delta\lambda &= \left(-48g^4 + 3\lambda^2\right) I_1 && (56)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_0 &= Z_\phi^{-2}(\lambda + \delta\lambda) && (57)\\
&= \lambda + \left(8g^2\lambda - 48g^4 + 3\lambda^2\right) I_1 && (58)\\
g_0 &= Z_\psi^{-1} Z_\phi^{-1/2}(g + \delta g) && (59)\\
&= g + 5g^3 I_1 && (60)
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \lambda_0 - \left(8g_0^2\lambda_0 - 48g_0^4 + 3\lambda_0^2\right) I_1 && (61)\\
g &= g_0 - 5g_0^3 I_1 && (62)
\end{aligned}
$$

The beta-functions:

$$
\begin{aligned}
\beta^\lambda &= \Lambda\frac{\partial}{\partial\Lambda}\lambda = \left(8g^2\lambda - 48g^4 + 3\lambda^2\right)\frac{1}{(4\pi)^2} && (63)\\
\beta^g &= \Lambda\frac{\partial}{\partial\Lambda}g = 5g^3\frac{1}{(4\pi)^2} && (64)
\end{aligned}
$$

For dimensional regularization, $g = g_r\Lambda^{(4-d)/2}$, $\lambda = \lambda_r\Lambda^{4-d}$,

$$
\begin{aligned}
\lambda_0 &= \Lambda^{4-d}\left[\lambda_r + \left(8g_r^2\lambda_r - 48g_r^4 + 3\lambda_r^2\right)\frac{1}{(4\pi)^{d/2}}\frac{1}{4-d}\right] && (65)\\
g_0 &= \Lambda^{(4-d)/2}\left[g_r + 5g_r^3\frac{1}{(4\pi)^{d/2}}\frac{1}{4-d}\right] && (66)
\end{aligned}
$$

The flows, writing $t = (4\pi)^2 t'$

$$\frac{d\lambda}{dt'} = -3\lambda^2 - 8g^2\lambda + 48g^4 \tag{67}$$

$$\frac{dg}{dt'} = -5g^3 \tag{68}$$

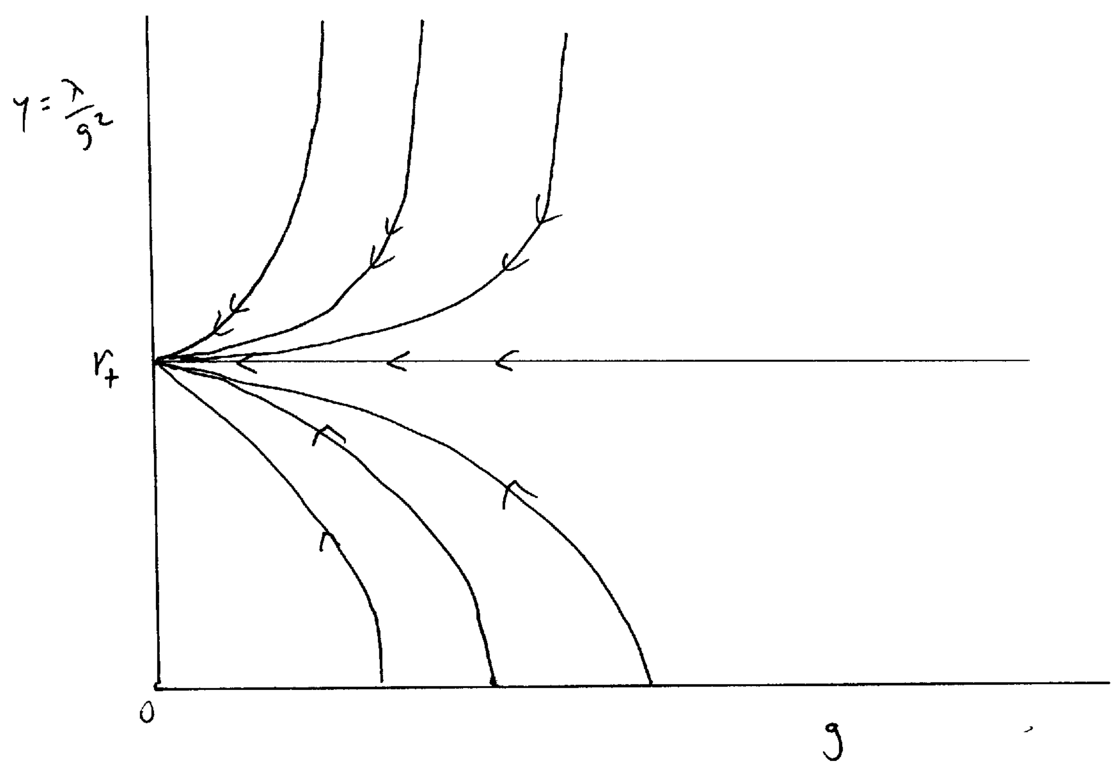
$$\frac{d\lambda}{dg} = \frac{g}{5}\left[3\left(\frac{\lambda}{g^2}\right)^2 + 8\left(\frac{\lambda}{g^2}\right) - 48\right] \tag{69}$$

$$y = \frac{\lambda}{g^2} \tag{70}$$

$$5g\frac{dy}{dg} = 3y^2 - 2y - 48 = 3(y - r_+)(y + r_-) \qquad r_\pm = \frac{1}{3}\left(\sqrt{145} \pm 1\right) \tag{71}$$

$$x = g^\alpha \qquad \alpha = \frac{2}{5}\sqrt{145} = \sqrt{23.2} \tag{72}$$

$$y = \frac{r_+ x_1 + r_- x}{x_1 - x} \tag{73}$$



12.2 Beta function of the Gross-Neveu model

From problem 11.3, the renormalized coupling constant is, to leading order in $1/N$,

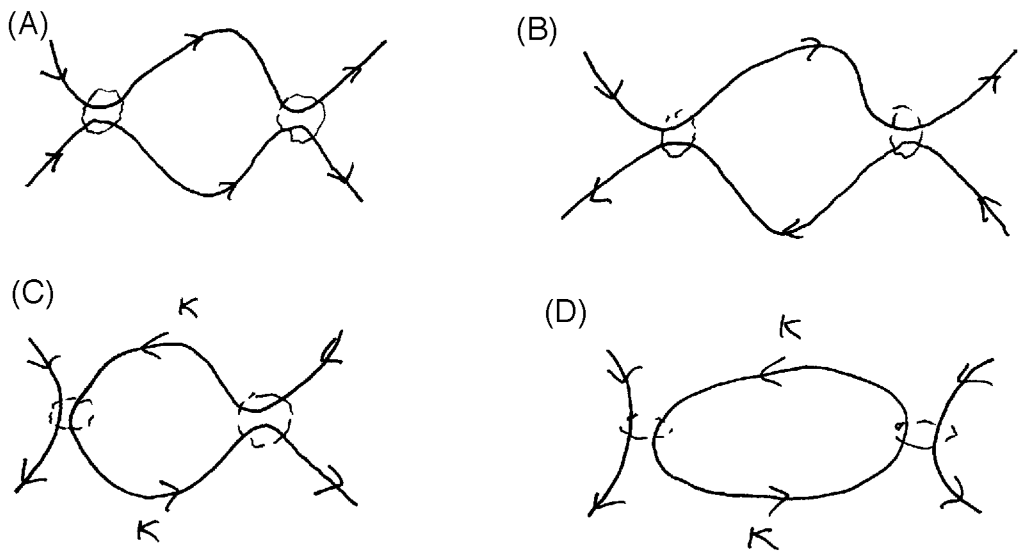
$$\frac{1}{g^2N} = \frac{1}{g_0^2N} + \frac{2}{2\pi} - \frac{1}{\pi}\ln\left(\frac{\Lambda_0}{\Lambda}\right) \tag{74}$$

so

$$\frac{-1}{(g^2N)^2}\Lambda\frac{\partial}{\partial\Lambda}g^2N = \frac{1}{\pi} \tag{75}$$

$$\beta(g^2N) = \Lambda\frac{\partial}{\partial\Lambda}g^2N = -\frac{1}{\pi}(g^2N)^2 \tag{76}$$

The Feynman diagram calculation is as follows. There are four 1-loop diagrams:



We only need the cutoff-dependent parts, so we evaluate these diagrams at zero external momenta.

Diagrams (A) and (B) individually vanish by the Majorana condition. That is, $\bar{\psi}\gamma^\mu\psi = \psi^T\gamma^0\gamma^\mu\psi = 0$ because the forms $\gamma^0\gamma^\mu$ are symmetric.

Diagram (D) gives (using euclidean Feynman rules, $1/2$ for the symmetry factor, (-1) for the fermion loop):

$$\begin{aligned}
(D) &= \int(\bar{\psi}\psi)^2\frac{1}{2}(-1)(-g^2)^2\int\frac{d^2k}{(2\pi)^2}\,\mathrm{tr}\left[\frac{1}{\not{k}}\frac{1}{\not{k}}\right] && (77)\\
&= \int\frac{1}{2}(\bar{\psi}\psi)^2(-g^4)\int\frac{d^2k}{(2\pi)^2}\,2N\frac{1}{(-|k|^2)} && (78)\\
&= \int\frac{1}{2}(\bar{\psi}\psi)^2g^4\int^{\Lambda_0}d|k|\frac{4\pi N}{(2\pi)^2}\,\frac{1}{|k|} && (79)\\
&= \int\frac{1}{2}(\bar{\psi}\psi)^2\,\frac{1}{\pi}g^4N\ln\left(\frac{\Lambda_0}{\Lambda}\right) && (80)
\end{aligned}$$

Diagram (C) is:

$$(C) = \int(\bar{\psi}\psi)(-g^2)^2\int\frac{d^2k}{(2\pi)^2}\,\bar{\psi}\frac{1}{\not{k}}\frac{1}{\not{k}}\psi \tag{81}$$

$$= \int (\bar{\psi}\psi)^2 g^4 \int \frac{d^2k}{(2\pi)^2} \frac{1}{(-|k|^2)} \tag{82}$$

$$= \int \frac{1}{2} (\bar{\psi}\psi)^2 (-\frac{1}{\pi} g^4) \ln \left(\frac{\Lambda_0}{\Lambda} \right) \tag{83}$$

so

$$(C) + (D) = \int \frac{1}{2} (\bar{\psi}\psi)^2 \frac{(N-1)g^4}{\pi} \ln \left(\frac{\Lambda_0}{\Lambda} \right) \tag{84}$$

$$\delta g^2 = -\frac{(N-1)g^4}{\pi} \ln \left(\frac{\Lambda_0}{\Lambda} \right) \tag{85}$$

$$g_0^2 N = g^2 N + \delta g^2 N = g^2 N - \frac{(N-1)Ng^4}{\pi} \ln \left(\frac{\Lambda_0}{\Lambda} \right) \tag{86}$$

$$\frac{1}{g^2 N} = \frac{1}{g_0^2 N} - \frac{1}{\pi} \left(1 - \frac{1}{N} \right) \ln \left(\frac{\Lambda_0}{\Lambda} \right) \tag{87}$$

$$\beta(g^2 N) = \Lambda \frac{\partial}{\partial \Lambda} g^2 N = -\frac{1}{\pi} \left(1 - \frac{1}{N} \right) (g^2 N)^2 \tag{88}$$

which agrees with problem 11.3 up to a $1/N$ correction. Also, this is consistent with the fact that the interaction vanishes exactly when $N = 1$.

Dimensional transmutation

The 2d Gross-Neveu model gives an example of dimensional transmutation. The RG says that the coupling constant g can be traded for a dimensionful parameter, say the fermion mass

$$m_F = \langle \sigma(x) \rangle = \Lambda e^{-\frac{\pi}{g^2 N}} . \tag{89}$$

All the correlation functions are completely determined, once m_F is fixed. The model is parametrized by this one dimensionful number.