Physics 616: Homework 2 Solution

P&S problem 9.1

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9.1 (a) Propagator and vertices for scalar QED

The euclidean action for scalar QED is (using the euclidean metric $\delta_{\mu\nu}$)

$$S_E = \int d^dx \left[\frac{1}{4}F^2 + (D_\mu\phi)^*(D^\mu\phi) + m^2\phi^*\phi\right] \tag{1}$$

$$= \int d^dx \left[\frac{1}{4}F^2 + \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi + eA_\mu(i\partial^\mu\phi^*\phi - i\phi^*\partial^\mu\phi) + e^2A_\mu A^\mu\phi^*\phi\right] \tag{2}$$

A quick way to get the propagator for the complex scalar is to write it in terms of two real scalars: $\phi = (\phi^1 + \phi^2)/\sqrt{2}$. The gaussian part of the action for ϕ becomes the ordinary action for two free real scalars

$$\int d^dx\, \frac{1}{2}\delta_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{2}\delta_{ij}m^2\phi^i\phi^j \tag{3}$$

so the ϕ, ϕ^* propagator is

$$\langle\, \phi^*(x_1)\, \phi(x_2)\, \rangle_{\text{gaussian}} = \frac{1}{2}\delta_{ij}\langle\, \phi^i(x_1)\, \phi^j(x_2)\, \rangle_{\text{gaussian}} \tag{4}$$

$$= \int \frac{d^dp}{(2\pi)^d}\, e^{ip(x_1-x_2)}\frac{1}{p_E^2 + m^2} \tag{5}$$

so the euclidean propagator is $1/(p_E^2 + m^2)$, the same as for the real scalar (though now the propagator comes marked with a direction, say from the ϕ^* end to the ϕ end. As for the real scalar, the Wick rotation gives $i/(p^2 - m^2 + i\epsilon)$, in the P&S convention for the Minkowski metric.

Alternatively, we can do the gaussian integral over complex variables $\phi^A, \bar{\phi}^{\bar{A}}$,

$$\int d\phi d\bar{\phi}\, e^{-\bar{\phi}^{\bar{A}}K_{\bar{A}B}\phi^B - j_A\phi^A - \bar{j}_{\bar{A}}\bar{\phi}^{\bar{A}}} = \det{}^{-1}(K)\, e^{j_A K^{A\bar{B}} j_{\bar{B}}} \tag{6}$$

so

$$\langle\, \phi^A\, \bar{\phi}^{\bar{B}}\, \rangle_{\text{gaussian}} = K^{A\bar{B}} \tag{7}$$

We read off the vertices from the euclidean action. The $A\phi^*\phi$ vertex is given by

$$-\int d^dx\, eA_\mu(i\partial^\mu\phi^*\phi - i\phi^*\partial^\mu\phi) = \int d^dx_1 d^dx_2 d^dx_3\, V^\mu(x_1, x_2, x_3)A_\mu(x_1)\phi^*(x_2)\phi(x_3) \tag{8}$$

$$V^\mu(x_1, x_2, x_3) = -e(i\partial^\mu_{(3)} - i\partial^\mu_{(2)})\int d^dx\, \delta^d(x_1 - x)\delta^d(x_2 - x)\delta^d(x_3 - x) \tag{9}$$

$$V^\mu(q, p, p') = \int d^dx_1 e^{iqx_1}\int d^dx_2 e^{-ipx_2}\int d^dx_3 e^{ip'x_3}V^\mu(x_1, x_2, x_3) \tag{10}$$

$$= (2\pi)^d\delta^d(q + p' - p)(-e)\delta^{\mu\nu}(p' + p)_\nu \tag{11}$$

The $AA\phi^*\phi$ vertex is given by

$$-\int d^dx\, e^2A_\mu A^\mu\phi^*\phi = \int d^dx_1 d^dx_2 d^dx_3 d^dx_4\, V^{\mu\nu}(x_1, x_2, x_3, x_4)\frac{1}{2}A_\mu(x_1)A_\nu(x_2)\phi^*(x_2)\phi(x_3) \tag{12}$$

$$V^{\mu\nu}(x_1, x_2, x_3, x_4) = -2e^2\delta^{\mu\nu}\int d^dx\, \delta^d(x_1 - x)\delta^d(x_2 - x)\delta^d(x_3 - x)\delta^d(x_4 - x) \tag{13}$$

$$V^{\mu\nu}(q_1, q_2, p, p') = \int d^dx_1 e^{iq_1x_1} \int d^dx_2 e^{iq_2x_2} \int d^dx_3 e^{-ipx_3} \int d^dx_4 e^{ip'x_4} V^{\mu\nu}(x_1, x_2, x_3, x_4) \tag{14}$$

$$= (2\pi)^d \delta^d(q_1 + q_2 + p' - p)(-2e^2\delta^{\mu\nu}) \tag{15}$$

9.1 (b) $e^+e^- \to \phi\phi^*$

The lowest order amplitude is as in P&S section 5.1 for $e^+e^- \to \mu^+\mu^-$, just replacing the photon-muon vertex with the photon-scalar vertex:

$$i\mathcal{M}\left(e^-(p)e^+(p') \to \phi^*(k)\phi(k')\right) = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(ie)(k-k')^\nu \tag{16}$$

Alternatively, use the reduction formula: write the lowest order diagram contributing to the connected correlation function $\langle\,\bar{\psi}\psi\phi^*\phi\,\rangle_c$, then amputate the external legs to get the amplitude.

The average over electron and positron polarizations works exactly the same as in P&S section 5.1:

$$\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{(q^2)^2}\mathrm{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right](k-k')_\mu(k-k')_\nu \tag{17}$$

$$= \frac{1}{4}\frac{e^4}{(q^2)^2}4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right](k-k')_\mu(k-k')_\nu \tag{18}$$

Neglecting m_e, this becomes

$$\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2 = \frac{e^4}{(q^2)^2}\left(2\left[p\cdot(k-k')\right]\left[p'\cdot(k-k')\right] - (p\cdot p')(k-k')^2\right) \tag{19}$$

In the center-of-mass frame,

$$p = (E, E\hat{z}) \tag{20}$$

$$p' = (E, -E\hat{z}) \tag{21}$$

$$k = (E, \vec{k}) \tag{22}$$

$$k' = (E, -\vec{k}) \tag{23}$$

$$\hat{z}\cdot\vec{k} = |\vec{k}|\cos\theta \tag{24}$$

$$E^2 - |\vec{k}|^2 = m^2 \tag{25}$$

$$q = (2E, \vec{0}) \tag{26}$$

$$q^2 = 4E^2 \tag{27}$$

$$k - k' = (0, 2\vec{k}) \tag{28}$$

$$(k-k')^2 = -4|\vec{k}|^2 \tag{29}$$

$$p\cdot(k-k') = -2E|\vec{k}|\cos\theta \tag{30}$$

$$p'\cdot(k-k') = 2E|\vec{k}|\cos\theta \tag{31}$$

$$p\cdot p' = 2E^2 \tag{32}$$

so

$$\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2 = \frac{e^4}{(4E^2)^2}\left(-8E^2|\vec{k}|^2\cos^2\theta + 8E^2|\vec{k}|^2\right) \tag{33}$$

$$= \frac{e^4|\vec{k}|^2}{2E^2}\sin^2\theta \tag{34}$$

$$= \frac{1}{2}e^4\left(1 - \frac{m^2}{E^2}\right)\sin^2\theta \tag{35}$$

The differential cross-section is then, as in P&S equation 5.12,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2E_{\rm cm}^2}\frac{|\vec{k}|}{16\pi^2 E_{\rm cm}}\cdot\frac{1}{4}\sum_{s,s'}|\mathcal{M}|^2 &(36)\\ &= \frac{\alpha^2}{8E_{\rm cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2}\sin^2\theta &(37)\end{aligned}$$

and the total cross-section is

$$\sigma_{total} = \frac{\pi\alpha^2}{3E_{\rm cm}^2}\left(1-\frac{m^2}{E^2}\right)^{3/2} \qquad (38)$$

Compare with $e^+e^- \to \mu^+\mu^-$ (P&S 5.12, 5.13):

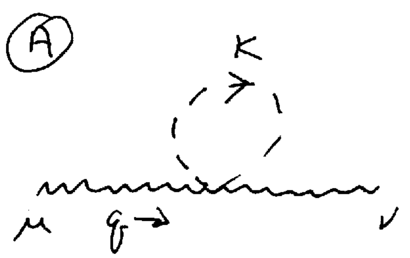
$$\begin{aligned}\frac{d\sigma}{d\Omega}(e^+e^-\to\mu^+\mu^-) &= \frac{\alpha^2}{4E_{\rm cm}^2}\left(1-\frac{m_\mu^2}{E^2}\right)^{1/2}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right] &(39)\\ \sigma_{total}(e^+e^-\to\mu^+\mu^-) &= \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\left(1-\frac{m_\mu^2}{E^2}\right)^{1/2}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right) &(40)\end{aligned}$$

Scalars prefer $\theta = 90°$ while muons prefer $\theta = 0°$. At high energy, the total cross-section for scalars is 1/4 that for muons. Near threshold, i.e., $E_{\rm cm} \approx 2m$ or $E_{\rm cm} \approx 2m_\mu$ respectively, the total cross-section for scalars grows much more slowly with energy than the total cross-section for muons.

9.1 (c) Contribution of the charged scalar to the photon vacuum polarization.

I do the calculation in euclidean spacetime, using the euclidean metric $g^E_{\mu\nu} = \delta_{\mu\nu}$ to raise and lower indices. I omit writing the 'E' subscript until the end, when I write the euclidean metric explicitly, and then Wick rotate to the Minkowski metric with the P&S signature: $g^E_{\mu\nu} = -g^{PS}_{\mu\nu}$.

There are two 1-loop diagrams:



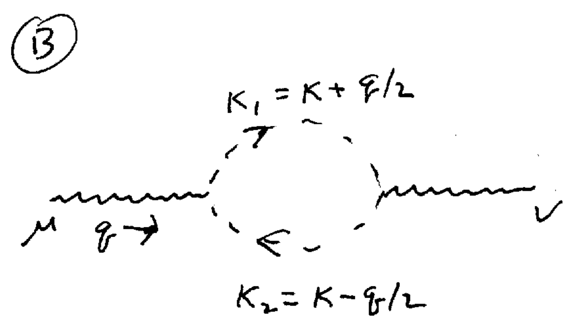


Diagram A contributes

$$\Pi^{\mu\nu}_{(A)} = \int\frac{d^dk}{(2\pi)^d}\frac{-2e^2\delta^{\mu\nu}}{k^2+m^2}\,. \qquad (41)$$

Diagram B contributes

$$\begin{aligned}\Pi^{\mu\nu}_{(B)} &= \int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}(2\pi)^d\delta^d(q-k_1+k_2)\frac{(-e)(k_1+k_2)^\mu(-e)(k_1+k_2)^\nu}{(k_1^2+m^2)(kp_2^2+m^2)} &(42)\\ &= \int\frac{d^dk}{(2\pi)^d}\frac{4e^2k^\mu k^\nu}{\left[\left(k+\frac{1}{2}q\right)^2+m^2\right]\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]} &(43)\end{aligned}$$

Recall that the euclidean Feynman rules give a minus sign for each vertex. Neither diagram has any symmetries, so the factor of 1 over the order of the symmetry group is just 1.

In the following, I will freely change integration variables in momentum integrals, making translations in momentum space, because all the dimensionally regularized momentum integrals are convergent.

Following the hint in the textbook, write

$$\Pi^{\mu\nu}_{(A)} = \int \frac{d^dk}{(2\pi)^d} \frac{-e^2\delta^{\mu\nu}\left[\left(k+\frac{1}{2}q\right)^2+m^2\right] - e^2\delta^{\mu\nu}\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]}{\left[\left(k+\frac{1}{2}q\right)^2+m^2\right]\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]} \tag{44}$$

$$= \int \frac{d^dk}{(2\pi)^d} \frac{-2e^2\left(k^2+\frac{1}{4}q^2+m^2\right)\delta^{\mu\nu}}{\left[\left(k+\frac{1}{2}q\right)^2+m^2\right]\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]} . \tag{45}$$

Now combine the two diagrams:

$$\Pi^{\mu\nu} = \Pi^{\mu\nu}_{(A)} + \Pi^{\mu\nu}_{(B)} = 2e^2 \int \frac{d^dk}{(2\pi)^d} \frac{2k^\mu k^\nu - \left(k^2+\frac{1}{4}q^2+m^2\right)\delta^{\mu\nu}}{\left[\left(k+\frac{1}{2}q\right)^2+m^2\right]\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]} \tag{46}$$

Introduce a Feynman parameter:

$$\frac{1}{\left[\left(k+\frac{1}{2}q\right)^2+m^2\right]\left[\left(k-\frac{1}{2}q\right)^2+m^2\right]} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{\delta(1-x_1-x_2)}{\left[k^2+(x_1-x_2)q\cdot k+\frac{1}{4}q^2+m^2\right]^2} \tag{47}$$

$$= \int_0^1 dx_1 \int_0^1 dx_2 \frac{\delta(1-x_1-x_2)}{\left[\tilde{k}^2-\frac{1}{4}(x_1-x_2)^2q^2+\frac{1}{4}q^2+m^2\right]^2} \tag{48}$$

$$= \int_0^1 dx \frac{1}{(\tilde{k}^2+\Delta)^2} \tag{49}$$

with

$$\tilde{k}^\mu = k^\mu + \left(x-\frac{1}{2}\right)q^\mu \qquad \Delta = x(1-x)q^2+m^2 . \tag{50}$$

Then

$$\Pi^{\mu\nu} = 2e^2 \int_0^1 dx \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{2k^\mu k^\nu - \left(k^2+\frac{1}{4}q^2+m^2\right)\delta^{\mu\nu}}{(\tilde{k}^2+\Delta)^2} \tag{51}$$

Substitute for k^μ in the numerator, then use euclidean rotational invariance of the integral over $\tilde{k}^\mu$ to get

$$\Pi^{\mu\nu} = 2e^2 \int_0^1 dx \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{2d^{-1}\tilde{k}^2\delta^{\mu\nu} + 2\left(x-\frac{1}{2}\right)^2 q^\mu q^\nu - \left[\tilde{k}^2+\left(x-\frac{1}{2}\right)^2q^2+\frac{1}{4}q^2+m^2\right]\delta^{\mu\nu}}{(\tilde{k}^2+\Delta)^2} \tag{52}$$

Break this up into a gauge invariant piece plus a piece proportional to $\delta^{\mu\nu}$:

$$\Pi^{\mu\nu} = 2e^2 \int_0^1 dx \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{2\left(x-\frac{1}{2}\right)^2(q^\mu q^\nu - \delta^{\mu\nu}q^2)}{(\tilde{k}^2+\Delta)^2} + 2e^2\delta^{\mu\nu}\int_0^1 dx\, R \tag{53}$$

where

$$R = \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{2\left(x-\frac{1}{2}\right)^2q^2 + 2d^{-1}\tilde{k}^2 - \left[\tilde{k}^2+\left(x-\frac{1}{2}\right)^2q^2+\frac{1}{4}q^2+m^2\right]}{(\tilde{k}^2+\Delta)^2} \tag{54}$$

This evaluates to

$$R = 0 \tag{55}$$

using the formula (in the cases $n = 1, 2$)

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \Delta^{\frac{d}{2}-n} . \tag{56}$$

So now we have

$$\Pi^{\mu\nu} = (\delta^{\mu\nu} q^2 - q^\mu q^\nu)\Pi \tag{57}$$

with

$$\begin{aligned}\Pi &= -4e^2 \int_0^1 dx\, \left(x - \frac{1}{2}\right)^2 \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{1}{(\tilde{k}^2 + \Delta)^2} &(58)\\ &= -4e^2 \int_0^1 dx\, \left(x - \frac{1}{2}\right)^2 \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Gamma(2)} \Delta^{\frac{d}{2}-2} &(59)\end{aligned}$$

The observable vacuum polarization is

$$\hat{\Pi} = \Pi - \Pi_{/q=0} \tag{60}$$

which has a finite limit as $\epsilon = 4 - d \to 0$,

$$\begin{aligned}\hat{\Pi} &= -4e^2 \int_0^1 dx\, \left(x - \frac{1}{2}\right)^2 \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{\epsilon}{2}\right) \left[\Delta^{-\frac{\epsilon}{2}} - (m^2)^{-\frac{\epsilon}{2}}\right] &(61)\\ &= -\frac{\alpha}{\pi} \int_0^1 dx\, \left(x - \frac{1}{2}\right)^2 \ln\left(\frac{m^2}{x(1-x)q_E^2 + m^2}\right) &(62)\end{aligned}$$

Where we now explicitly indicate that we have been using the euclidean space-time metric. To Wick rotate, replace q_E^2 by $-q^2$, where q^2 is now the Minkowski metric in the P&S convention:

$$\hat{\Pi}(q^2) = -\frac{\alpha}{\pi} \int_0^1 dx\, \left(x - \frac{1}{2}\right)^2 \ln\left(\frac{m^2}{-x(1-x)q^2 + m^2}\right) \tag{63}$$

In the limit $q_E^2 = -q^2 \gg m^2$, this becomes

$$\hat{\Pi} \to -\frac{\alpha}{12\pi} \ln\left(\frac{m^2}{-q^2}\right) \tag{64}$$

The $O(\alpha)$ vacuum polarization due to electrons is given by P&S equation 7.91,

$$\hat{\Pi}_e = -\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \ln\left(\frac{m^2}{-x(1-x)q^2 + m^2}\right) \tag{65}$$

which has the limit

$$\hat{\Pi}_e \to -\frac{\alpha}{3\pi} \ln\left(\frac{m^2}{-q^2}\right) \tag{66}$$

which is 4 times larger than that due to a charged scalar.