Physics 616: Homework 1 Solution

P&S problem 9.2

Daniel Friedan

February 4, 2008

9.2 (a) I did this in class. Let $\tau_f = \tau_i + \beta$.

$$\langle q_f | e^{-(\tau_f - \tau_i)H} | q_i \rangle = \int_{\substack{q(\tau_i) = q_i \\ q(\tau_f) = q_f}} \mathcal{D}q \, e^{-S^E[q]} \tag{1}$$

$$\begin{aligned} \mathrm{tr}(e^{-\beta H}) &= \int d^N q_i \, \langle q_i | e^{-(\tau_f - \tau_i)H} | q_i \rangle && (2) \\ &= \int d^N q_i \int_{\substack{q(\tau_i) = q_i \\ q(\tau_f) = q_i}} \mathcal{D}q \, e^{-S^E[q]} && (3) \\ &= \int_{q(\tau_i) = q(\tau_i + \beta)} \mathcal{D}q \, e^{-S^E[q]} && (4) \end{aligned}$$

functional integral for $x(\tau)$ in periodic euclidean time, ignoring possibly divergent constants independent of ω:

$$x(\tau) = \sum_{n=-\infty}^{\infty} x_n \beta^{-1/2} e^{2\pi i n \tau/\beta} \tag{5}$$

$$x_n^* = x_{-n} \tag{6}$$

$$\int_0^\beta d\tau \, x(\tau)^2 = \sum_{n=-\infty}^{\infty} x_{-n} x_n \tag{7}$$

$$\mathcal{D}x = (const) \, dx_0 \prod_{n=1}^{\infty} d(\mathbf{Re}\, x_n) d(\mathbf{Im}\, x_n) \tag{8}$$

$$\begin{aligned} S^E[x] &= \int_0^\beta d\tau \left[\frac{1}{2} (\partial_\tau x)^2 + \frac{1}{2} \omega^2 x^2 \right] && (9) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2} x_{-n} [\Delta_n + \omega^2] x_n && (10) \end{aligned}$$

$$\Delta_n = \left(\frac{2\pi n}{\beta} \right)^2 \tag{11}$$

$$\begin{aligned} \mathrm{tr}\left(e^{-\beta H}\right) &= \int \mathcal{D}x \, e^{-S^E[x]} && (12) \\ &= (const) \int dx_0 \, e^{-\frac{1}{2}\omega^2 x_0^2} \prod_{n=1}^{\infty} d(\mathbf{Re}\, x_n) d(\mathbf{Im}\, x_n) \, e^{-x_{-n}[\Delta_n + \omega^2] x_n} && (13) \\ &= (const) \, \omega^{-1} \prod_{n=1}^{\infty} [\Delta_n + \omega^2]^{-1} && (14) \end{aligned}$$

$$= \quad (const)\,\omega^{-1}\prod_{n=1}^{\infty}\left[1+\Delta_n^{-1}\omega^2\right]^{-1} \tag{15}$$

$$= \quad (const)\left(\frac{\beta\omega}{2}\right)^{-1}\prod_{n=1}^{\infty}\left[1+\left(\frac{\beta\omega}{2}\right)^2\frac{1}{\pi^2 n^2}\right]^{-1} \tag{16}$$

$$= \quad (const)\,\frac{1}{\sinh(\beta\omega/2)} \tag{17}$$

$$= \quad (const)\sum_{n=0}^{\infty} e^{-\beta(n\omega+\omega/2)} \tag{18}$$

which is the partition function of the harmonic oscillator (with ground state energy $\omega/2$).

Note that we have to cut off the product over Fourier modes labelled by n until step (15). At that point, a divergent constant independent of ω has been pulled out, rendering the product finite.

Additional note

It is not that difficult to calculate the overall constant in the partition function of the harmonic oscillator, from the path integral.

Cut off the path integral by integrating over discrete paths with time step $\epsilon = \beta/M$. The discrete Fourier modes x_n are labelled by $n = 0, 1, \cdots M-1$. The time derivative ∂_τ is approximated by the finite difference operator. The Fourier modes are the eigenfunctions, with eigenvalues

$$\partial_\tau e^{2\pi i n\tau/\beta} = \epsilon^{-1}\left(e^{2\pi i n\epsilon/\beta}-1\right)e^{2\pi i n\tau/\beta} \tag{19}$$

so

$$\partial_\tau^* \partial_\tau \partial_\tau e^{2\pi i n\tau/\beta} = \Delta_n e^{2\pi i n\tau/\beta}\,. \tag{20}$$

with

$$\Delta_n = \epsilon^{-2}\left|e^{2\pi i n\epsilon/\beta}-1\right|^2 = \frac{4}{\epsilon^2}\sin^2\left(\frac{\pi n\epsilon}{\beta}\right) = \frac{4}{\epsilon^2}\sin^2\left(\frac{\pi n}{M}\right)\,. \tag{21}$$

The euclidean path integral integral is

$$Z = \int \mathcal{D}x\, e^{-S^E[x]} \tag{22}$$

with

$$\mathcal{D}x = \prod_\tau (2\pi\epsilon)^{-1/2} dx(\tau) \tag{23}$$

and

$$S^E[x] = \epsilon\sum_\tau\left(\frac{1}{2}\partial_\tau x\partial_\tau x + \frac{1}{2}\omega^2 x^2\right) \tag{24}$$

so

$$Z = (2\pi\epsilon)^{-M/2}\det{}^{-1/2}\left[\frac{\epsilon}{2\pi}(\partial_\tau^*\partial_\tau+\omega^2)\right] = \det{}^{-1/2}\left[\epsilon^2(\partial_\tau^*\partial_\tau+\omega^2)\right] \tag{25}$$

(Note the factor ϵ inside the determinant, which comes from writing $S^E[x]$ as a sum over discrete τ.) Now calculate

$$Z \quad = \quad \epsilon^{-M}\prod_{n=0}^{M-1}(\Delta_n+\omega^2)^{-1/2} \tag{26}$$

$$= \quad \epsilon^{-M}\omega^{-1}\prod_{n=1}^{M-1}\Delta_n^{-1/2}\prod_{n=1}^{M-1}(1+\Delta_n^{-1}\omega^2)^{-1/2} \tag{27}$$

$$= \quad \epsilon^{-M}\omega^{-1}\prod_{n=1}^{M-1}\left[\frac{2}{\epsilon}\sin\left(\frac{\pi n}{M}\right)\right]^{-1}\prod_{n=1}^{M-1}(1+\Delta_n^{-1}\omega^2)^{-1/2} \tag{28}$$

Use the identity (e.g., from GR 1.392 1.)

$$M = \prod_{n=1}^{M-1} 2\sin\left(\frac{\pi n}{M}\right) \tag{29}$$

to calculate

$$\begin{aligned} Z &= \omega^{-1}\epsilon^{-1}M^{-1}\prod_{n=1}^{M-1}(1+\Delta_n^{-1}\omega^2)^{-1/2} && (30)\\ &= (\beta\omega)^{-1}\prod_{n=1}^{M-1}(1+\Delta_n^{-1}\omega^2)^{-1/2} && (31)\end{aligned}$$

Now take the limit $M \to \infty$ to obtain

$$Z = \frac{1}{2\sinh(\beta\omega/2)} = \sum_{n=0}^{\infty} e^{-\beta(n\omega+\omega/2)}\,. \tag{32}$$

Note that our only approximation was in the original formulation of the discrete path integral, where we discarded $O(\epsilon^2)$ error terms in each discrete time step, coming from commutators of the kinetic term in the hamiltonian with the potential term. The present calculation can be regarded as confirmation that we were justified in discarding those terms.

9.2 (c) Partition function of a free scalar field.

We've already done the basic work:

$$\mathrm{tr}(e^{-\beta H}) = \int_{q(\tau_i)=q(\tau_i+\beta)} \mathcal{D}q\, e^{-S^E[q]} \tag{33}$$

where

$$A \text{ is } \vec{x} \qquad q^A \text{ is } \phi(\vec{x},0) \qquad q^A(\tau) \text{ is } \phi(\vec{x},\tau) = \phi(x_E) \tag{34}$$

$$S^E[q] = \int_0^\beta dx^4 \int d^3\vec{x}\,\left(\frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2\right) \tag{35}$$

so, again,

$$\begin{aligned} \mathrm{tr}(e^{-\beta H}) &= (const)\ \det{}^{-1/2}\left(-g_E^{\mu\nu}\partial_\mu\partial_\nu + m^2\right) && (36)\\ &= (const)\ \det{}^{-1/2}\left(-\partial_\tau^2 - \partial_i\partial^i + m^2\right) && (37)\\ & && (38)\end{aligned}$$

Now expand $\phi(x_E)$ in Fourier modes:

$$\sum_{n=-\infty}^{\infty} \phi_{n,\vec{k}}\beta^{-1/2}e^{2\pi i n\tau/\beta + i\vec{k}_i\vec{x}^i} \tag{39}$$

The eigenvalues of $-g_E^{\mu\nu}\partial_\mu\partial_\nu + m^2$ are

$$\Delta_{n,\vec{k}} = \left(\frac{2\pi n}{\beta}\right)^2 + E_{\vec{k}}^2 \tag{40}$$

$$E_{\vec{k}} = \left(\vec{k}^2 + m^2\right)^{1/2} \tag{41}$$

so

$$\mathrm{tr}(e^{-\beta H}) = (const)\ \prod_{n,\vec{k}} \Delta_{n,\vec{k}}^{-1/2} \tag{42}$$

For this to make sense, to make the product finite, we need cutoffs for euclidean time and space. For example, we can discretize euclean time, with time step ϵ, and latticize space, as a periodic cubic lattice with lattice spacing a and finite volume L^3. The derivatives all become finite difference operators. Now we have a finite collection of $(L/a)^3$ independent harmonic oscillators, indexed by discrete $\vec{k}$,

$$k_i = \frac{2\pi n_i}{L} \qquad |n_i| \leq \frac{L}{2a} \tag{43}$$

The eigenvalues of $-g_E^{\mu\nu}\partial_\mu\partial_\nu + m^2$ now are

$$\Delta_{n,\vec{k}} = \left(\frac{2\pi n}{\beta}\right)^2 + E_{0,\vec{k}}^2 \tag{44}$$

where

$$E_{0,\vec{k}} = \left(\sum_{i=1}^{3}\left[\frac{2}{a}\sin\left(\frac{ak_i}{2}\right)\right]^2 + m^2\right)^{1/2} \tag{45}$$

are the energies of the harmonic oscillators.

We can use the previous result to get, in the limit $\epsilon \to 0$,

$$\begin{aligned}
\mathrm{tr}(e^{-\beta H}) &= \prod_{\vec{k}} \frac{1}{2\sinh(\beta E_{0,\vec{k}}/2)} &(46)\\
&= \prod_{\vec{k}}\sum_{n=0}^{\infty} e^{-\beta(nE_{0,\vec{k}}+E_{0,\vec{k}}/2)} &(47)\\
&= \left(\prod_{\vec{k}} e^{-\beta E_{0,\vec{k}}/2}\right)\prod_{\vec{k}}\sum_{n=0}^{\infty} e^{-\beta nE_{0,\vec{k}}} &(48)
\end{aligned}$$

We shift H by the sum of ground state energies, so the energy of the vacuum becomes 0. This cancels the first product on the rhs, giving:

$$\mathrm{tr}(e^{-\beta H}) = \prod_{\vec{k}}\sum_{n=0}^{\infty} e^{-\beta nE_{0,\vec{k}}} \tag{49}$$

Now we can take the limit $a \to 0$ to obtain

$$\mathrm{tr}(e^{-\beta H}) = \prod_{\vec{k}}\sum_{n=0}^{\infty} e^{-\beta nE_{\vec{k}}} \tag{50}$$

which is the partition function for relativistic scalar particles in a periodic cubic box of volume L^3.

To take the limit $L \to \infty$,

$$\begin{aligned}
\mathrm{tr}(e^{-\beta H}) &= e^{\sum_{\vec{k}} -\ln\left[2\sinh(\beta E_{\vec{k}}/2)\right]} &(51)\\
&= e^{\sum_{\vec{k}} -\beta E_{\vec{k}}/2} e^{\sum_{\vec{k}} -\ln\left[1-e^{-\beta E_{\vec{k}}}\right]} &(52)
\end{aligned}$$

The first exponential is the divergent contribution of the ground state energies of all the oscillators, which we have gotten rid of by shifting H. The $\vec{k}$ range over a periodic lattice with lattice spacing $2\pi/L$, so we can approximate, for large L,

$$\mathrm{tr}(e^{-\beta H}) = e^{L^3(2\pi)^{-3}\int d^3\vec{k}\, -\ln\left[1-e^{-\beta E_{\vec{k}}}\right]} \tag{53}$$

What has a limit as $L \to \infty$ is

$$L^{-3}\ln\mathrm{tr}(e^{-\beta H}) \tag{54}$$

the free energy *density*.

9.2 (d) Fermionic harmonic oscillator.

$$S_E = \int_0^\beta d\tau\, \bar{\psi} \qquad \psi(\tau+\beta) = -\psi(\tau)(\partial_\tau + \omega)\psi \tag{55}$$

Here is a quick argument that, in the path integral for the partition function, the Grassmann paths $\psi(\tau)$ should be anti-periodic. Suppose $\beta > \tau > 0$.

$$\begin{aligned}
\text{tr}\, e^{-\beta H} T\left\{\bar{\psi}(\tau)\psi(0)\right\} &= \text{tr}\, e^{-\beta H}\bar{\psi}(\tau)\psi(0) & (56)\\
&= \text{tr}\, \bar{\psi}(\tau)\psi(0)e^{-\beta H} & (57)\\
&= \text{tr}\, \bar{\psi}(\tau)e^{-\beta H}\psi(\beta) & (58)\\
&= \text{tr}\, e^{-\beta H}\psi(\beta)\bar{\psi}(0) & (59)\\
&= \text{tr}\, e^{-\beta H} T\left\{\psi(\beta)\bar{\psi}(0)\right\} & (60)\\
&= \text{tr}\, e^{-\beta H} T\left\{-\bar{\psi}(0)\psi(\beta)\right\} & (61)
\end{aligned}$$

So we can expand:

$$\psi = \beta^{-1/2} \sum_{n\in\frac{1}{2}+\mathbf{Z}} \psi_n e^{2\pi i n\tau/beta} \tag{62}$$

$$S_E = \sum_{n\in\frac{1}{2}+\mathbf{Z}} \bar{\psi}_n d_n \psi_n \tag{63}$$

$$d_n = \frac{2\pi i n}{\beta} + \omega \tag{64}$$

The functional integral becomes, up to an overall constant,

$$\begin{aligned}
\text{tr}\, e^{-\beta H} &= (const) \prod \sum_{n\in\frac{1}{2}+\mathbf{Z}} d_n & (65)\\
&= (const) \prod \sum_{n\in\frac{1}{2}+\mathbf{Z}} \left(1 + \frac{\beta\omega}{2\pi i n}\right) & (66)\\
&= (const)\ \cosh(\beta\omega) & (67)\\
&= (const)\ \left(e^{-\beta\omega/2} + e^{\beta\omega/2}\right) & (68)
\end{aligned}$$

This is the partition function of the 2-state system with energies $\pm\omega/2$, up to an overall constant independent of ω.

9.2 (e) Partition function of the photon field.

After the now familiar manipulations to evaluate the gaussian functional integral for the partion function of the photon field, in the ξ-gauge, we get (using the euclidean signature spacetime index convention):

$$\text{tr}\, e^{-\beta H} = (const)\ \det\left(-\partial^2\right) \det{}^{-1/2}\left(-\partial^2\delta^\mu_\nu + (1-\xi)\partial^\mu\partial_\nu\right) \tag{69}$$

with periodic boundary conditions in euclidean time. The first determinant is the Fadeev-Popov determinant, and the second is the result of the gaussian functional integral over the 4-component photon field.

Expanding in Fourier modes gives

$$\begin{aligned}
\text{tr}\, e^{-\beta H} &= (const) \prod_k k^2 \det{}^{-1/2}\left[k^2\delta^\mu_\nu - (1-\xi)k^\mu k_\nu\right] & (70)\\
&= (const) \prod_k k^2 \left[(k^2)^3(\xi k^2)\right]^{-1/2} & (71)\\
&= (const) \prod_k (k^2)^{-1} & (72)
\end{aligned}$$

Comparing to the result for the free scalar field, this is the partition function for two free massless bosons: i.e., the two photon polarizations. (Note that the ξ-dependence in the functional determinant is cancelled by the ξ dependence in the original constant, coming from the gauge fixing process.)