Physics 616

Note on the Ricci tensor of S^{N-1}

Daniel Friedan

May 3, 2008

Here is a derivation of the Ricci tensor of the unit sphere S^{N-1} in $\mathbb{R}^N$:

$$R_{IJ} = (N-2) g_{IJ} \,. \tag{1}$$

The unit sphere in $\mathbb{R}^N$:

$$S^{N-1} = \left\{ n^I \in \mathbb{R}^N ,\; n^2 = 1 \right\} \tag{2}$$

The projection on the tangent hyperplane:

$$P^I_J = \delta^I_J - \frac{1}{n^2} n^I n_J \tag{3}$$

The round metric:

$$g_{IJ} = P_{IJ} = \delta_{IJ} - \frac{1}{n^2} n_I n_J \tag{4}$$

The covariant derivative, for $Pv = v$, $Pu = u$:

$$\nabla_v u = P \partial_v u = P(v^K \partial_K u) \tag{5}$$

(It is easily verified that $\nabla g = 0$, and ∇ is torsion-free, so ∇ is the unique torsion-free covariant derivative that preserves the metric.)

The curvature tensor, for $Pv = v$, $Pw = w$, $Pu = u$:

$$\begin{aligned} [\nabla_v, \nabla_w] u &= P\partial_v (P \partial_w u) - P \partial_w (P \partial_v u) && (6)\\ &= P(\partial_v P) \partial_w (Pu) - P(\partial_w P)\partial_v (Pu) + P[\partial_v, \partial_w] u && (7)\\ &= P[\partial_v P,\, \partial_w P] Pu + P \partial_{\partial_v w - \partial_w v} u && (8) \end{aligned}$$

so

$$\begin{aligned} v^K w^L R^I_{JKL} &= P[\partial_v P,\, \partial_w P]^I_J && (9)\\ R^I_{JKL} &= P[\partial_K P,\, \partial_L P]^I_J && (10) \end{aligned}$$

Calculate:

$$\partial_K P = P(\partial_K P)(1-P) + (1-P)(\partial_K P)P \tag{11}$$

$$\begin{aligned} \partial_K P^I_J &= 2 n_K n^I n_J - \delta^I_K n_J - n^I \delta_{JK} && (12)\\ (P\partial_K P)^I_J &= -P^I_K n_J && (13)\\ (\partial_L P P)^I_J &= -n^I P_{JL} && (14)\\ (P\partial_K P \partial_L P P)^I_J &= P^I_K P_{JL} && (15) \end{aligned}$$

so

$$\begin{aligned} R^I_{JKL} &= P^I_K P_{JL} - P^I_L P_{JK} && (16)\\ R_{JL} &= R^K_{JKL} = (N-1) P_{JL} - P_{JL} && (17) \end{aligned}$$

so

$$R_{IJ} = (N-2) g_{IJ} \tag{18}$$