Physics 616

Note on C-Z equation for correlation functions

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Z is the generating functional for the renormalized correlation functions:

$$Z = \langle\, e^{-\int j_r \phi_r} \,\rangle \tag{1}$$

Where $j_r(x)$ is the naively dimensionless source and $\phi_r(x)$ is the naively dimensionless field:

$$j_r(x) = \Lambda^{-(d+2)/2} j(x) \tag{2}$$

$$\phi_r(x) = \Lambda^{-(d-2)/2} \phi(x) \tag{3}$$

$$\int j_r \phi_r = \int d^d x\, \Lambda^d\, j_r(x) \phi_r(x) \tag{4}$$

The RG equation is

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \partial_i - \left(\frac{d+2}{2} - \gamma \right) j_r \frac{\partial}{\partial j_r} \right] \; Z(\Lambda, \lambda_r, j_r) = 0 \tag{5}$$

The λ_r^i are the macroscopic dimensionless coupling constants (besides the source), e.g. λ and $m_r^2 = \Lambda^{-2} m^2$ in the ϕ^4 theory.

Expand in j_r. Schematically:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \partial_i - \left(\frac{d+2}{2} - \gamma \right) j_r \frac{\partial}{\partial j_r} \right] \; (\int j_r \phi_r)^n = 0 \tag{6}$$

This holds for arbitrary sources $j_r(x)$, so we have the C-Z equation

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \partial_i + nd - \left(\frac{d+2}{2} - \gamma \right) \right] \langle\, \phi_r(x_1) \cdots \phi_r(x_n) \,\rangle = 0 \tag{7}$$

To get the C-Z equation for the Fourier transformed dimensionless correlation functions, write

$$\int j_r \phi_r = \int d^d x\; \Lambda^d j_r(x) \phi_r(x) \tag{8}$$

$$= \int d^d x\; \Lambda^d j_r(x) \int \frac{d^d p}{(2\pi)^d}\; \Lambda^{-d} e^{ipx} \tilde{\phi}_r(p) \tag{9}$$

$$= \iint \frac{d^d x\, d^d p\, e^{ipx}}{(2\pi)^d}\; j_r(x) \tilde{\phi}_r(p) \tag{10}$$

Note that the scale Λ does not appear in the coupling between $j_r(x)$ and $\tilde{\phi}_r(p)$. So

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \partial_i - \left(\frac{d+2}{2} - \gamma \right) j_r \frac{\partial}{\partial j_r} \right] \; (\iint j_r \tilde{\phi}_r)^n = 0 \tag{11}$$

gives

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \partial_i - n \left(\frac{d+2}{2} - \gamma \right) \right] \langle\, \tilde{\phi}_r(p_1) \cdots \tilde{\phi}_r(p_n) \,\rangle = 0 \tag{12}$$

This is the form of the C-Z equation I find easiest to use: the rg equation on the dimensionless correlation functions.

Derive the dimensionless 2-point function of the free massless scalar from the C-Z equation:

$$\langle\, \tilde{\phi}_r(p_1)\tilde{\phi}_r(p_2)\,\rangle = \delta^d(\Lambda^{-1}\sum_i p_i)\, g_r(\Lambda^{-2}p_1^2) = \Lambda^d\delta^d(\sum_i p_i)\, g_r(\Lambda^{-2}p_1^2) \tag{13}$$

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - 2\left(\frac{d+2}{2}\right)\right]\Lambda^d\delta^d(\sum_i p_i)\, g_r(\Lambda^{-2}p_1^2) = 0 \tag{14}$$

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - 2\right] g_r(\Lambda^{-2}p_1^2) = 0 \tag{15}$$

so

$$g_r(\Lambda^{-2}p_1^2) = (const)\frac{1}{\Lambda^{-2}p_1^2} \tag{16}$$

which is correct.

In general we write

$$\langle\, \tilde{\phi}_r(p_1)\cdots\tilde{\phi}_r(p_n)\,\rangle = \delta^d\left(\sum_i \Lambda^{-1}p_i\right)\; G_r(p_1,\ldots,p_n) \tag{17}$$

and we have the C-Z equation in the form

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta^i\partial_i + d - n\left(\frac{d+2}{2} - \gamma\right)\right]\; G_r(p_1,\ldots,p_n) = 0 \tag{18}$$

To go to the dimensionful fields:

$$\begin{aligned}
\tilde{\phi}_r(p) &= \int d^dx\,\Lambda^d e^{ipx}\phi_r(x) &(19)\\
&= \int d^dx\,\Lambda^d e^{ipx}\Lambda^{-(d-2)/2}\phi(x) &(20)\\
&= \Lambda^{(d+2)/2}\tilde{\phi}(p) &(21)
\end{aligned}$$

so

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta^i\partial_i + n\left(\frac{d+2}{2}\right) - n\left(\frac{d+2}{2} - \gamma\right)\right]\langle\, \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\,\rangle = 0 \tag{22}$$

or

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta^i\partial_i + n\gamma\right]\langle\, \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\,\rangle = 0 \tag{23}$$

We can easily use this form of the C-Z equation to derive, for example, the massless 2-point function. But I find the derivation using the dimensionless correlation functions more transparent.