

The beta function $B(x, y)$

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The Beta function is defined, for $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, as

$$B(\alpha, \beta) = \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1}.$$

To evaluate this, consider

$$\begin{aligned} \Gamma(\alpha + \beta)B(\alpha, \beta) &= \int_0^\infty t^{\alpha+\beta-1} e^{-t} dt \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} \\ &= \int_0^\infty t^{\beta-1} e^{-t} dt \int_0^t du u^{\alpha-1} \left(1 - \frac{u}{t}\right)^{\beta-1} \\ &= \int_0^\infty e^{-t} dt \int_0^t du u^{\alpha-1} (t-u)^{\beta-1} \\ &= \int_0^\infty du u^{\alpha-1} \int_u^\infty e^{-t} (t-u)^{\beta-1} dt \\ &= \int_0^\infty du u^{\alpha-1} e^{-u} \int_0^\infty e^{-v} v^{\beta-1} dv \\ &= \Gamma(\alpha)\Gamma(\beta), \end{aligned}$$

so

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

Other interesting forms:

By writing $x = \sin^2 \theta$ we have

$$B(\alpha, \beta) = 2 \int_0^{\pi/2} (\sin \theta)^{2\alpha-1} (\cos \theta)^{2\beta-1} d\theta.$$

By writing $x = 1/(1+u)$ we have

$$B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du.$$

$\Gamma(z)$ has a pole of residue 1 at $z = 0$, but it is important to have the finite term as well. For infinitesimal ϵ we have

$$\Gamma(1 + \epsilon) = \int_0^\infty t^\epsilon e^{-t} dt \approx \int_0^\infty (1 + \epsilon \ln t) e^{-t} dt = 1 - \epsilon\gamma,$$

where

$$\gamma = - \int_0^{\infty} (\ln t) e^{-t} dt = 0.57721\ 56649 \dots$$

is known as the Euler's constant or Mascheroni's constant and can also be written as

$$\gamma = \lim_{m \rightarrow \infty} \left(\sum_{j=1}^m \frac{1}{j} - \ln m \right).$$

Thus as $z \rightarrow 0$,

$$\Gamma(z) = z^{-1} \Gamma(1+z) = \frac{1}{z} - \gamma + \mathcal{O}(z).$$

There is lots more about Gamma functions in Whittaker and Watson, Modern Analysis, chapter 12.