

Energy in Bremsstrahlung, Note on p. 179

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From homework #2 we learned that the energy density is the $(0, 0)$ component of the Noether current associated with translations. Generally, associated with $\delta x^\mu = g^{\mu\nu}$ and $\delta A_\mu = 0$, we have

$$T^{\nu\mu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho} (\partial_\sigma A_\rho \delta x^\sigma - \delta A_\rho) - \mathcal{L}\delta x^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho} \partial_\sigma A_\rho g^{\sigma\nu} - \mathcal{L}\delta g^{\mu\nu}.$$

From homework #1 we have

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

so

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\rho} = F^{\rho\mu}.$$

Thus

$$T^{\nu\mu} = F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma},$$

and

$$\mathcal{H} = T^{00} = \frac{1}{4}(F^{\rho\mu}F_{\rho\mu} - 2F^{0\mu}F_{0\mu}) = \frac{1}{2}(F^{0j}F_{0j} - F^{ij}F_{ij}) = \frac{1}{2}(E^2 + B^2).$$

Note that we can write \mathcal{H} in the strange fashion

$$\mathcal{H} = -\frac{1}{4}\sum_\rho F^{\rho\mu}F_{\mu\rho},$$

where I have explicitly given the \sum_ρ , because we don't have a summation convention on two upper indices, but the \sum_μ is left implicit.

As discussed on pages 178-179 in Peskin and Schroeder, the radiation field in the impulse approximation for electron scattering is given by the residues of the poles at $k^0 = \pm|\vec{k}|$ in

$$A^\mu(x) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot x} \frac{-ie}{k^2} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)$$

The residues of $1/k^2 = 1/[(k^0 - |\vec{k}|)(k^0 + |\vec{k}|)]$ are $\pm(1/2|\vec{k}|)$ at $k^0 = \pm|\vec{k}|$ respectively, so

$$A_{\text{rad}}^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \left\{ \left(\frac{p'^\mu}{k^0 E' - \vec{k} \cdot \vec{p}'} - \frac{p^\mu}{k^0 E - \vec{k} \cdot \vec{p}} \right) e^{-i(k^0 t - \vec{k} \cdot \vec{x})} - \left(\frac{p'^\mu}{-k^0 E' - \vec{k} \cdot \vec{p}'} - \frac{p^\mu}{-k^0 E - \vec{k} \cdot \vec{p}} \right) e^{-i(-k^0 t - \vec{k} \cdot \vec{x})} \right\} \Big|_{k^0 = \pm|\vec{k}|}$$

Reversing the sign of the integration variable \vec{k} in the second term, this may be written as

$$\begin{aligned} A_{\text{rad}}^\mu(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \left\{ \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right) e^{-ik \cdot x} + \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right) e^{ik \cdot x} \right\} \Big|_{k^0 = \pm|\vec{k}|} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} \left\{ \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right) e^{-ik \cdot x} + c.c. \right\} \Big|_{k^0 = \pm|\vec{k}|} \end{aligned}$$

in agreement with 6.6 of Peskin and Schroeder. Thus we have

$$A_{\text{rad}}^\mu(x) = \int \frac{d^3k}{2(2\pi)^3} \mathcal{A}^\mu(\vec{k}) (e^{-ik \cdot x} + e^{ik \cdot x}), \quad \text{with } \mathcal{A}^\mu(\vec{k}) = \frac{-e}{|\vec{k}|} \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right).$$

Note $k_\mu \mathcal{A}^\mu(\vec{k}) = 0$ for $k^0 = |\vec{k}|$. If we define $\bar{k}^\mu := (-|\vec{k}|, \vec{k})$ (so $\bar{k}_\mu = -\bar{k}^\mu$), we also have $\bar{k}_\mu \mathcal{A}^\mu(-\vec{k}) = 0$. This will prove useful in evaluating the energy in the radiation field.

From the expression for $A_{\text{rad}}^\mu(x)$ we have

$$F_{\text{rad}}^{\mu\nu}(x) = -i \int \frac{d^3k}{2(2\pi)^3} (k^\mu \mathcal{A}^\nu(\vec{k}) - k^\nu \mathcal{A}^\mu(\vec{k})) (e^{-ik \cdot x} - e^{ik \cdot x}) \Big|_{k^0 = |\vec{k}|}.$$

so

$$\begin{aligned} H &= -\frac{1}{4} \int d^3x \sum_\rho F_{\text{rad}}^{\rho\mu} F_{\text{rad}\ \mu}^\rho \\ &= \frac{1}{4} \int d^3x \int \frac{d^3k}{2(2\pi)^3} \int \frac{d^3k'}{2(2\pi)^3} \\ &\quad \sum_\rho (k^\rho \mathcal{A}^\mu(\vec{k}) - k^\mu \mathcal{A}^\rho(\vec{k})) (e^{-ik \cdot x} - e^{ik \cdot x}) \\ &\quad \times (k'^\rho \mathcal{A}_\mu(\vec{k}') - k'^\mu \mathcal{A}_\rho(\vec{k}')) (e^{-ik' \cdot x} - e^{ik' \cdot x}) \Big|_{\substack{k^0 = |\vec{k}| \\ k'^0 = |\vec{k}'|}} \end{aligned}$$

We can use $\int d^3x d^3k' e^{i(\vec{k}\pm\vec{k}')\cdot\vec{x}} = (2\pi)^3$ with $\mp\vec{k}$ substituted for \vec{k}' in the rest of the expression. As all k^0 and k'^0 are positive, it would be better to say $k' \rightarrow -\vec{k}$ in the first case.

Thus we have

$$H = \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} \sum_{\rho} \left\{ -2 \left(k^{\rho} \mathcal{A}^{\mu}(\vec{k}) - k^{\mu} \mathcal{A}^{\rho}(\vec{k}) \right) \left(k^{\rho} \mathcal{A}_{\mu}(\vec{k}) - k^{\mu} \mathcal{A}_{\rho}(\vec{k}) \right) \right. \\ \left. + \left(k^{\rho} \mathcal{A}^{\mu}(\vec{k}) - k^{\mu} \mathcal{A}^{\rho}(\vec{k}) \right) \left(\bar{k}^{\rho} \mathcal{A}_{\mu}(-\vec{k}) - \bar{k}^{\mu} \mathcal{A}_{\rho}(-\vec{k}) \right) \right. \\ \left. \times \left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t} \right) \right\} \Big|_{k^0=|\vec{k}|}.$$

where $\bar{k} = (-k^0, \vec{k})$. Note $\sum_{\rho} k^{\rho} V^{\rho} = -\bar{k} \cdot V$, and $\sum_{\rho} k^{\rho} k^{\rho} = 2|\vec{k}|^2$, so

$$H = \frac{1}{16} \int \frac{d^3k}{(2\pi)^3} \left\{ -8|\vec{k}|^2 \mathcal{A}^2(\vec{k}) + 4k \cdot \mathcal{A}(\vec{k}) \bar{k} \cdot \mathcal{A}(\vec{k}) \right. \\ \left. + \left(-2k^2 \mathcal{A}(\vec{k}) \cdot \mathcal{A}(-\vec{k}) + 2k \cdot \mathcal{A}(-\vec{k}) k \cdot \mathcal{A}(\vec{k}) \right) \left(e^{2i|\vec{k}|t} + e^{-2i|\vec{k}|t} \right) \right\} \\ = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} |\vec{k}|^2 \mathcal{A}^2(\vec{k})$$

where we have used $k^2 = 0$ and $k \cdot \mathcal{A}(\vec{k}) = 0$. Thus we have

$$H = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} |\vec{k}|^2 \left(\frac{-e}{|\vec{k}|} \right)^2 \left(\frac{2p \cdot p'}{k \cdot p' k \cdot p} - \frac{m^2}{(k \cdot p')^2} - \frac{m^2}{(k \cdot p)^2} \right) \\ = \frac{e^2}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{2p \cdot p'}{k \cdot p' k \cdot p} - \frac{m^2}{(k \cdot p')^2} - \frac{m^2}{(k \cdot p)^2} \right)$$

in agreement with 6.13.

On getting 6.17:

For extreme relativistic \vec{v} and \vec{v}' , the integral over the angles of \hat{k} is dominated by the region around each of the two \vec{v} 's. With θ the angle between \hat{k} and \vec{v} , we will integrate from $\theta = 0$, but we don't want to get too close the region near \vec{v}' , as we wish to approximate $1 - \hat{k} \cdot \vec{v}' \approx 1 - \vec{v} \cdot \vec{v}'$. So we will go halfway, up to $\cos \theta = 1 - \frac{1}{2}(1 - \vec{v} \cdot \vec{v}')$ and write $I(\vec{v}, \vec{v}') \approx \frac{1}{|\vec{v}|} \ln(1 - vu) \Big|_{\frac{1+\vec{v}\cdot\vec{v}'}{2}} = \frac{1}{|\vec{v}|} \left(\ln \frac{1}{2} + \ln \frac{1-\vec{v}\cdot\vec{v}'}{1-|\vec{v}|} \right)$, plus the similar contribution from $\vec{v} \leftrightarrow \vec{v}'$. This justifies 6.17 in the ultrarelativistic limit.