An extra note on Noether’s Theorem

I was asked to clarify the connection of conserved charges emerging as a consequence of a symmetry and the generator of those symmetry transformations.

Let us consider the simpler case of a discrete dynamical system, with coordinates \( q_i \), with a Lagrangian \( L(q_i, \dot{q}_i, t) \) which is invariant under an infinitesimal symmetry transformation \( q_i \rightarrow q_i + \epsilon_i(q, t) \), so that the change produced in the Lagrangian is

\[
\delta L = \sum_i \left[ \frac{\partial L}{\partial q_i} \epsilon_i(q, t) + \frac{\partial L}{\partial \dot{q}_i} \frac{d\epsilon_i(q, t)}{dt} \right]
\]

where the time derivative of \( \epsilon \) is a stream derivative including the variation of all of the \( q \)'s,

\[
\frac{d\epsilon_i(q, t)}{dt} = \frac{\partial \epsilon_i(q, t)}{\partial t} + \sum_j \frac{\partial \epsilon_i(q, t)}{\partial q_j} \dot{q}_j.
\]

Using the equations of motion on the first term in \( \delta L \), we have

\[
\delta L = \sum_i \left[ \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \epsilon_i(q, t) + \frac{\partial L}{\partial \dot{q}_i} \frac{d\epsilon_i(q, t)}{dt} \right] = \frac{d}{dt} \left( \sum_i \frac{\partial L}{\partial \dot{q}_i} \epsilon_i(q, t) \right).
\]

Thus if the change \( \delta L \) is zero,

\[
Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \epsilon_i(q, t) = \sum_i P_i \epsilon_i(q, t)
\]

is conserved. Here \( P_i \) is the canonical momentum conjugate to \( q_i \).

But we see that the quantum mechanical commutator or classical Poisson bracket generates (minus) the infinitesimal symmetry transformation,

\[
[Q, q_i] = \sum_j P_j \epsilon_j(q, t), q_i = \sum_j [P_j, q_i] \epsilon_j(q, t) = \sum_j [-\delta_{ij}] \epsilon_j(q, t)
\]

\[
= -\epsilon_i(q, t) = -\delta q_i
\]
In Field Theory

In field theory the situation is similar but complicated by our consideration of changes in the $x^\mu$ as well as in the fields. We considered an infinitesimal variation

$$x_\mu \rightarrow x'_\mu = x_\mu + \delta x_\mu$$

with the fields varying by

$$\phi'_i(x') = \phi_i(x) + \delta \phi_i(x; \phi_k(x)) = \phi_i(x') + \delta \phi_i.$$  (2)

The two forms $\delta \phi$ and $\delta x$ are useful in differing expressions for $\delta L$, which we defined by

$$\delta L(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x') = L(\phi'_i(x'), \partial'_\mu \phi'_i(x'), x') - L(\phi_i(x), \partial_\mu \phi_i(x), x) \left| \frac{\partial x'^\nu}{\partial x^\mu} \right|. \quad (3)$$

We found that the current for this infinitesimal transformation

$$\epsilon J^\mu = - \frac{\partial L}{\partial \partial_\mu \phi_i} \delta \phi_i + \frac{\partial L}{\partial \phi_i} \partial_\nu \phi_i \delta x'^\nu - L \delta x'^\nu + \Lambda^\mu.$$  (3)

has a vanishing divergence if $\delta L = \partial_\mu \Lambda^\mu$.

Let us avoid some interpretation problems by assuming the transformation doesn’t change time $\delta t = 0$ so we have no problem defining a conserved charge as $Q = \int d^3x J^0(\vec{x})$, and let’s assume we don’t need a $\Lambda$. Then

$$Q = - \int d^3x \frac{\partial L}{\partial \phi_i}(\vec{x}) \left( \delta \phi_i(\vec{x}) + \delta x'^\nu \partial_\nu \phi_i(\vec{x}) \right) = - \int d^3x \pi_i(\vec{x}) b \phi_i(\vec{x}),$$

where $\pi_i(\vec{x})$ is the canonical momentum density conjugate to $\phi_i$. Then

$$[Q, \phi_j(\vec{y})] = - \int d^3x \left[ \pi_i(\vec{x}) b \phi_i(\vec{x}), \phi_j(\vec{y}) \right] = - \int d^3x \left[ \pi_i(\vec{x}), \phi_j(\vec{y}) \right] b \phi_i(\vec{x}) = - \int d^3x \left[ -i \delta^3(\vec{x} - \vec{y}) \delta_{ij} \right] b \phi_i(\vec{x}) = i b \phi_j(\vec{y}).$$

So $Q$ does generate the infinitesimal symmetry transformation at each point in space.