Noether’s Theorem, (condensed version)
Sept. 16, 2013

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The simplest field transforms like a scalar:

\[ \phi'(x') = \phi(x) \]

for a scalar field

but more generally

\[ \phi'_j(x') = \phi_j(x) + \delta \phi_j (x, \phi_k(x)) , \]

which is what you would need if \( \phi \) were a vector, with \( \delta \vec{o} = \delta \vec{\omega} \times \vec{o} \).

Note that I have defined \( \delta \vec{\phi} = \delta \vec{\omega} \times \vec{\phi} \).

We have long known that if a coordinate does not appear undifferentiated in the Lagrangian, it is an “ignorable” coordinate, and the conjugate momentum is conserved. If \( L(r, \vec{p}) \) does not depend on one coordinate \( x \), \( \partial L/\partial x \) is conserved, and if \( L(r, \theta, \phi, \cdots) \) does not depend on \( \phi \), \( \partial L/\partial \phi = L_z \) is conserved. But, if the potential depends only on \( \phi \), \( \delta \vec{\phi} \) should be a vector, with \( \delta \vec{\phi} \).

An ignorable coordinate \( q_i \) corresponds to a symmetry group \( q_i \rightarrow q_i + c \), where \( c \) is an arbitrary constant. We can view this group of symmetries as generated by and infinitesimal transformation \( q_i \rightarrow q_i + \delta q_i \). But there can be more complicated symmetries of the Lagrangian, for example \( r_i \rightarrow r_i + \delta r_i \), which is a rotation about \( \delta \vec{\omega} \), and the three generators generate the rotation group and insures the conservation of all three components of the angular momentum.

For fields, the possible symmetries may be more complicated yet, for the value of each point in space may be changed by the symmetry, possibly in a way that depends on fields at other points.

The simplest situation is for an “internal symmetry”, where the change of \( \phi_i(x^\mu) \) depends only on fields at \( x^\mu \),

\[ \phi_i(x^\mu) \rightarrow \phi_i(x^\mu) + \delta \phi_i (x^\mu; \phi_j(x^\mu)) . \]

To be very general, however, we might consider that \( \delta \phi_i(x^\mu) \) could depend on \( \phi_j(y^\mu) \) everywhere. This is, however, much too general, but we do want to consider the possibility that the symmetry changes the \( x^\mu \) coordinates, relating the fields at some new point \( x'^\mu \) to those at an old point \( x^\mu \), as for example what happens under rotation, where \( \vec{E}'(\vec{x}') \) is a vector rotated from \( \vec{E}(\vec{x}) \), not from \( \vec{E}(\vec{x}') \).

For an infinitesimal symmetry transformation, the new point \( x'^\mu \) will differ by an infinitesimal from \( x^\mu \):

\[ x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu . \]

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