Functional Integral for a Scalar Field.

We have seen that the time evolution operator in quantum mechanics can be understood as a functional integral

\[ \langle q_f, t_f | e^{-iH(t_f-t_i)} | q_i, t_i \rangle = \int Dq Dp \exp i \int dt [pq - H(q, p, t)], \]

where from now on \( \hbar = 1 \). This is quite general, so we are not restricted to a finite number of degrees of freedom \( q \). In particular, we could consider a scalar field with Hamiltonian

\[ H = \int d^3x \left( \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2}(\nabla \phi)^2(x) + V(\phi(\vec{x})) \right), \]

[For a free scalar field of mass \( m \), \( V(\phi) = \frac{1}{2}m^2 \phi^2 \). Then we need

\[ \int D\phi D\pi \exp i \int d^3x dt \left[ -\frac{1}{2}(\pi^2(x) + \pi(x)\phi(x)) \right] \exp i \int d^3x dt \left[ -\frac{1}{2}(\nabla \phi)^2 - V(\phi) \right]. \]

Only the first exponential depends on \( \pi(x) \), and we can treat each point of space-time independently. Using

\[ \int \frac{dx}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \lambda x^2 + Bx \right] = \frac{e^{B^2/2\lambda}}{\sqrt{\lambda}}, \]

with \( \lambda = i\Delta^2x \), and ignoring the field-independent normalization factors, we have

\[ \int D\pi \exp i \int d^3x dt \left[ -\frac{1}{2}(\pi^2(x) + \pi(x)\phi(x)) \right] \sim \exp i \int d^3x dt \frac{1}{2} \dot{\phi}^2, \]

and thus the functional integral becomes

\[ \int D\phi \exp i \int d^3x \left[ \frac{1}{2} \dot{\phi}^2(x) - \frac{1}{2}(\nabla \phi)^2 - V(\phi) \right] = \int D\phi \exp i \int d^3x L(\phi), \]

where the Lagrangian density

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi). \]

We now have a form of quantum mechanics based on the Lagrangian rather than the Hamiltonian, and except for the integration region (which goes from \( t_i \) to \( t_f \) in time but includes all of space) our formulation is Lorentz invariant. It will also have symmetries under any transformation which leaves the Lagrangian density \( L \) invariant.

Correlation Functions

In quantum mechanics, the expectation values or transitions of operators are given by \( \langle \psi_f | O_1 O_2 | \psi_i \rangle \), where we generally think of operators at a fixed time. But in field theory we need to think of operators acting at arbitrary points in space-time. In the Schrödinger picture, where the fundamental operators are considered time-independent but the states evolve, the evolution \( \phi_i \rightarrow \phi_f \) under the influence of some operators \( O(x^\mu)O(x^\nu) \) is given by

\[ \langle \phi_f | e^{-i(t_f-t_i)H} O(x^\mu)O(x^\nu) e^{-i(t_0-x_i^0)H} O(x^\mu) e^{-i(t_0-x_i^0)H} | \phi_i \rangle, \]

where I have assumed \( x_2 \) is later than \( x_1 \). We may rewrite this in the Heisenberg picture, in which operators evolve while the states are fixed. The connection is

\[ O_H(\vec{x}, t) = e^{iHt} O_S(\vec{x}) e^{-iHt}. \]

Then the transition amplitude is

\[ \langle \phi_f | e^{-it_f H} O_H(x^\mu) O_H(x^\nu) e^{it_H} | \phi_i \rangle. \]

Consider the operators to be the fields themselves. Returning to the Schrödinger picture, we see that

\[ \langle \phi_f | e^{-it_f H} \phi_H(x^\mu) \phi_H(x^\nu) e^{it_H} | \phi_i \rangle \]

\[ = \langle \phi_f | e^{-i(t_f-t_i)H} \phi_S(x^\mu) e^{-i(x^\mu_0-x^\mu_i)H} \phi_S(x^\nu) e^{-i(x^\nu_0-x^\nu_i)H} | \phi_i \rangle \]

\[ = \int_{t_2} D\phi_2(\vec{x}) \ U(\phi_f, \phi_2, t_f, t_2) \phi_2(x^\mu_2) \int_{t_1} D\phi_1(\vec{x}) \ U(\phi_2, \phi_1, t_2, t_1) \phi_1(x^\mu_1) U(\phi_1, \phi_i, t_1, t_i) \]

\[ = \int D\phi \phi(x^\mu) \phi(x^\nu) \exp i \int_{t_i}^{t_f} d^4x L(\phi). \]

Note that the \( \int D\phi \) in the next-to-last line are integrals only over states at a given time, while the \( \int D\phi \) in the last line is a functional integral over space-time.

The connection of the operator expectation value to the functional integral assumed \( x^\mu_2 > x^\mu_i \). The functional integral itself is unchanged under \( x_2 \leftrightarrow x_1 \), as the \( \phi \)'s there are commuting c-numbers. But in the operator form \( \phi_H(x^\mu_2) \) and \( \phi_H(x^\mu_i) \) need not commute, and our expression is only correct if the later one occurs first. Therefore introduce the “Time ordering operator” \( T \), which tells us to order any operators appearing in its scope in decreasing order of their times. This is not an operator on the Hilbert space;
rather it is a metaoperator which acts on our symbolic expressions, reordering them before they are allowed to physically act on the Hilbert space. Our corrected expression is thus

$$\langle \phi_f | e^{-it^H T} (\phi_H(x_2^\mu) \phi_H(x_1^\mu)) e^{it^H} | \phi_i \rangle = \int \mathcal{D}\phi \phi(x_2^\mu) \phi(x_1^\mu) \exp i \int_{t_i}^{t_f} d^4x \mathcal{L}(\phi).$$

We are interested in evaluating correlation functions, which are the expected values of operator products in the vacuum, or lowest energy state. If we insert a complete set of energy eigenstates $\sum_n |n\rangle \langle n| = 1$ at the ends of our operator, we have

$$\int \mathcal{D}\phi \phi_2(x_2^\mu) \phi_1(x_1^\mu) \exp i \int_{t_i}^{t_f} d^4x \mathcal{L}(\phi) = \sum_n \sum_m \langle \phi_f | n\rangle \langle n| e^{-it^f E_n T} (\phi_H(x_2^\mu) \phi_H(x_1^\mu)) e^{it^i E_m} | m\rangle \langle m| \phi_i \rangle.$$

Now we can formally extract the vacuum matrix elements by formally letting $t_f$ approach $+\infty$ with a negative imaginary part, $t_f \to \infty(1 - i\epsilon)$. If we call the lowest state $|\Omega\rangle$ and assume its energy is 0, all other states get a contribution $\exp(-(E_n \epsilon \cdot \infty)) \to 0$, and we extract only the vacuum state from the sum. Similarly we let $t_i \to -\infty(1 - i\epsilon)$. Assume the states $|\phi_f\rangle$ and $|\phi_i\rangle$ have some overlap with the vacuum state $|\Omega\rangle$. Then

$$\langle \Omega | T\phi_H(x_2) \phi_H(x_1) | \Omega \rangle = \frac{\langle \Omega | T\phi_H(x_2) \phi_H(x_1) | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \lim_{T \to \infty(1 - i\epsilon)} \frac{\int \mathcal{D}\phi \phi_2(x_2^\mu) \phi_1(x_1^\mu) \exp i \int_{-T}^{T} d^4x \mathcal{L}(\phi)}{\int \mathcal{D}\phi \exp i \int_{-T}^{T} d^4x \mathcal{L}(\phi)}.$$