Physics 615 Fall, 2007
Homework Solution #9

1 [10 pts] Find the spin-averaged cross section for an electron scattering off a very heavy Dirac particle of charge e. Do not assume $m_e \approx 0$, but do take the limit that the other mass M goes to infinity. You can use the expression just above 5.60 with the cross section formula 4.84, working in the center of mass, which is the same as the lab frame in the $M \to \infty$ limit.

Solution 1 We want to evaluate the average over initial spins and sum over final spins of the differential cross section. The electron's energy and momentum are $(E, \vec{p})$ initially and $(E, \vec{p}\,')$ in the final state, while the heavy particle has energy M, and momentum $(-\vec{p})$ which is negligible compared to M. The electron's velocity $v = \beta = p/E$, while the heavy particle's velocity is ≈ 0. The spin averaged cross section is therefore

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{2E\,2M\,(p/E)} \frac{p}{(2\pi)^2 4(M+E)} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2.$$

In the expression above 5.60 for the sum of $|\mathcal{M}|^2$, the second trace becomes

$$M^2 \operatorname{Tr}\left[(\gamma^0+1)\gamma_\mu(\gamma^0+1)\gamma_\nu\right] = 4M^2(g_{\mu\nu} + \delta^0_\mu\delta^0_\nu - g_{\mu\nu} + \delta^0_\mu\delta^0_\nu) = 8M^2\delta^0_\mu\delta^0_\nu.$$

In the middle expression, the first $g_{\mu\nu}$ comes from the two 1's, the rest from $\gamma^0\gamma_\mu\gamma^0\gamma_\nu$, without any contribution from one 1 and one γ^0 because that is the trace of an odd number of γ's.

The first trace is

$$\operatorname{Tr}\left[(\not{p}' + m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] = 4[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p'_\rho p^\rho - m_e^2)],$$

just following the result above 5.10. With the initial momentum along the z axis, $\vec{p} = (0,0,p)$ and $\vec{p}\,' = (p\sin\theta, 0, p\cos\theta)$, so $p'_\rho p^\rho = E^2 - p^2\cos\theta$. Then the first trace, with $\mu = \nu = 0$, becomes $4(E^2 + p^2\cos\theta + m_e^2)$. Also note that $q^\mu = p'^\mu - p^\mu = (0, p\sin\theta, 0, p(\cos\theta - 1))$, so

$$q^2 = -p^2(\sin^2\theta + (\cos\theta - 1)^2) = -2p^2(1-\cos\theta) = -4p^2\sin^2\frac{\theta}{2}.$$

Putting this all together,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4(q^2)^2}32M^2\left(2E^2 - p^2(1-\cos\theta)\right)$$

and

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} &= \frac{1}{4(4\pi)^2}\frac{e^4}{64p^4\sin^4\frac{\theta}{2}}32\left(2E^2 - p^2(1-\cos\theta)\right) \\ &= \frac{\alpha^2}{4p^2\beta^2\sin^4\frac{\theta}{2}}\left(1-\beta^2\sin^2\frac{\theta}{2}\right).\end{aligned}$$

2 [5 pts] Write the amplitude for Compton scattering

$$e^-_{\vec{p},s} + \gamma_{\vec{k},\lambda} \longrightarrow e^-_{\vec{p}\,',s'} + \gamma_{\vec{k}\,',\lambda'}$$

for an electron of spin s and photon of polarization λ, as

$$i\mathcal{M} = i\epsilon^{\lambda'\,*}_{\mu}(k')\epsilon^{\lambda}_{\nu}(k)\mathcal{M}^{\mu\nu},$$

where

$$i\mathcal{M}^{\mu\nu} = -ie^2\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p),$$

as given in 5.74. Show explicitly what the Ward identity claims must be true, namely that $k_\nu\mathcal{M}^{\mu\nu} = 0$.

Solution 2

$$\begin{aligned}k_\nu\mathcal{M}^{\mu\nu} &= -e^2\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\not{k} + 2\gamma^\mu p\cdot k}{2p\cdot k} + \frac{-\not{k}\not{k}'\gamma^\mu + 2\not{k}p^\mu}{-2p\cdot k'}\right]u(p) \quad (1)\\ &= -e^2\bar{u}(p')\left[0 + \gamma^\mu + \frac{\not{k}\not{k}'\gamma^\mu}{2p\cdot k'} - \frac{\not{k}p^\mu}{p\cdot k'}\right]u(p) \quad (2)\\ &= -e^2\bar{u}(p')\left[\gamma^\mu + \frac{\not{k}(\not{p}+\not{k}-\not{p}')\gamma^\mu}{2p\cdot k'} - \frac{\not{k}p^\mu}{p\cdot k'}\right]u(p) \quad (3)\\ &= -e^2\bar{u}(p')\left[\gamma^\mu + \frac{2\not{k}p^\mu - \not{k}\gamma^\mu\not{p} + 0 - 2k\cdot p'\gamma^\mu + \not{p}'\not{k}\gamma^\mu}{2p\cdot k'} - \frac{\not{k}p^\mu}{p\cdot k'}\right]\end{aligned}$$

$$u(p) \qquad (4)$$

$$= -e^2\bar{u}(p')\left[\gamma^\mu + \frac{2\not{k}p^\mu - m\not{k}\gamma^\mu - 2k\cdot p'\gamma^\mu + m\not{k}\gamma^\mu}{2p\cdot k'} - \frac{\not{k}p^\mu}{p\cdot k'}\right]$$

$$u(p) \qquad (5)$$

$$= -e^2\bar{u}(p')\left[\gamma^\mu + \frac{\not{k}p^\mu}{p\cdot k'} - \frac{k\cdot p'\gamma^\mu}{p\cdot k'} - \frac{\not{k}p^\mu}{p\cdot k'}\right]u(p) \qquad (6)$$

$$= 0 \qquad (7)$$

In (2) we used $\not{k}^2 = k^2 = 0$, and in (3) we used momentum conservation to substitute $p + k - p'$ for k'. In (4) we anticommuted $\not{p}$ to the right to act on $u(p)$, and $\not{p}'$ to the left to act on $\bar{u}(p')$, and in (5) we made use of $\not{p}u(p) = m\,u(p)$ and $\bar{u}(p')\not{p}' = m\,\bar{u}(p')$. Finally, if we square what momentum conservation tells us, $p - k' = p' - k$, we find $p\cdot k' = p'\cdot k$, so everything cancels.