Physics 615, 2007 Homework Sol. #2

We will repeatedly be using

$$\delta\mathcal{L} = \Big(\partial_\mu \delta x^\mu\Big)\mathcal{L} + \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + (\delta\partial_\mu\phi_i)\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} + \delta x^\mu \frac{\delta\mathcal{L}}{\delta x^\mu} \tag{1}$$

to see if the transformation is a symmetry, $\delta\mathcal{L} = \partial_\mu\Lambda^\mu$. If so, we can define the conserved current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i}\left(\partial_\nu\phi_i\frac{d\delta x^\nu}{d\epsilon} - \frac{d\delta\phi_i}{d\epsilon}\right) - \mathcal{L}\frac{d\delta x^\mu}{d\epsilon} + \Lambda^\mu. \tag{2}$$

1) With $\delta x^\mu = 0$, $\delta\phi_i = A_{ij}\phi_j$, and $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi^2$, we have

$$\frac{\partial\mathcal{L}}{\partial\phi_i} = -m^2\phi_i\,, \qquad \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} = \partial^\mu\phi_i\,, \qquad \frac{\delta\mathcal{L}}{\delta x^\mu} = 0.$$

Thus

$$\begin{aligned}\delta\mathcal{L} &= \sum_i\left(\sum_j A_{ij}\phi_j\right)\Big(-m^2\phi_i\Big) + \sum_i\left(\sum_j A_{ij}\partial_\mu\phi_j\right)(\partial^\mu\phi_i)\\ &= \sum_{ij} A_{ij}\times(\text{something symmetric under } i\leftrightarrow j) = 0.\end{aligned}$$

Thus we have a symmetry with $\Lambda^\mu = 0$, and

$$J^\mu = \partial^\mu\phi_i A_{ij}\phi_j, \qquad Q = \int J^0 d^3x = \sum_{ij} A_{ij}\dot{\phi}_i\phi_j.$$

2) On homework #1, we found that we could reexpress the Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi^2$, in terms of $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and its conjugate $\phi^\dagger$, as

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi.$$

The internal symmetry transformation on the original fields $\delta\phi_1 = \epsilon\phi_2$, $\delta\phi_2 = -\epsilon\phi_1$, implies

$$\delta\phi = \frac{\epsilon\phi_2 - i\epsilon\phi_1}{\sqrt{2}} = -i\epsilon\phi, \qquad \delta\phi^\dagger = \frac{\epsilon\phi_2 + i\epsilon\phi_1}{\sqrt{2}} = i\epsilon\phi^\dagger,$$

which is just the infinitesimal version of $\phi \rightarrow e^{-i\epsilon}\phi$. We already know it is a symmetry — in fact it is trivial that each term in $\mathcal{L}$ is unchanged. The current

$$J^\mu = (\partial^\mu\phi_1)\phi_2 - (\partial^\mu\phi_2)\phi_1 = i\phi^\dagger\overleftrightarrow{\partial^\mu}\phi =: i\left(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right).$$

[Note we are treating the fields classically, not worrying about commuting them.] The conserved charge is therefore

$$\begin{aligned} Q &= \int d^3x J^0(x) = \int d^3x \left(\phi^\dagger(x)\dot{\phi}(x) - \dot{\phi}^\dagger(x)\phi(x)\right) \\ &= \int d^3x \left(\phi^\dagger(x)\pi(x) - \pi^\dagger(x)\phi(x)\right). \end{aligned}$$

3) With $\delta x^\mu = \epsilon a^\mu$, $\delta\phi_i = 0$, we have none of the four terms in (1) giving any contribution, so again $\delta\mathcal{L} = 0 = \Lambda^\mu$. The current

$$J^\mu = \left((\partial^\mu\phi_i)(\partial_\nu\phi_i) - \mathcal{L}\delta^\mu_\nu\right)a^\nu = a_\nu T^{\nu\mu},$$

where the conserved energy-momentum tensor is

$$T^{\nu\mu} = (\partial^\mu\phi_i)((\partial^\nu\phi_i) - \frac{1}{2}g^{\nu\mu}\left((\partial^\rho\phi_i)(\partial_\rho\phi_i) - m^2\phi^2\right), \qquad \text{so} \quad \partial_\mu T^{\nu\mu} = 0.$$

The conserved charges are the total momenta,

$$P^\nu = \int d^3x \left[\dot{\phi}_i\partial^\nu\phi_i - \frac{1}{2}g^{0\nu}\left((\partial^\rho\phi_i)(\partial_\rho\phi_i) - m^2\phi^2\right)\right].$$

For $\nu = j \neq 0$, this is simply

$$P^j = \int d^3x\dot{\phi}_i\partial^j\phi_i \quad \text{or} \quad \vec{P} = -\int d^3x\dot{\phi}_i\vec{\nabla}\phi_i,$$

while for the zero component we have

$$P^0 = H = \int d^3x\mathcal{H}, \quad \text{where}$$

$$\mathcal{H} = \dot{\phi}_i\dot{\phi}_i - \frac{1}{2}(\partial^\rho\phi_i)(\partial_\rho\phi_i) + \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left[(\dot{\phi})^2 + (\vec{\nabla}\phi_i)^2 + m^2\phi^2\right]$$

is the Hamiltonian density.

4) Again $\delta\phi_i = 0$, but $\delta x^\mu = \epsilon L^\mu{}_\nu x^\nu$. As the stream derivative of $\mathcal{L}$ is zero, the only possible contribution to $\delta\mathcal{L}$ is the one with a factor of

$$\partial_\mu \delta x^\mu = \epsilon L^\mu{}_\nu \delta^\nu_\mu = \epsilon L_{\mu\nu} g^{\nu\mu} = 0,$$

because L is antisymmetric while the metric g is symmetric. Thus we again have a symmetry, with current

$$J^\mu = \frac{1}{2} L_{\nu\rho} \mathcal{M}^{\mu\nu\rho} = \frac{1}{2}\left((\partial^\mu\phi_i)(\partial_\nu\phi_i) - \mathcal{L}\delta^\mu_\nu\right) L^\nu{}_\rho x^\rho = \frac{1}{2} T^\mu{}_\nu L^\nu{}_\rho x^\rho,$$

so $\mathcal{M}^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu$, with the explicit antisymmetrization necessary because $L_{\nu\rho}$ is an arbitrary *antisymmetric* matrix.

5) With $\delta x^\mu = \epsilon x^\mu$ the infinitesimal form of a scale transformation, we might expect ϕ_i to scale as well, with some dimension D, or $\phi_i'(x') = (1+\epsilon)^{-D}\phi_i(x)$, which is infinitesimally $\delta\phi_i = -D\epsilon\phi_i$, and as

$$\frac{\partial}{\partial x'^\mu}\phi_i(x') = (1+\epsilon)^{-1}\frac{\partial}{\partial x^\mu}(1+\epsilon)^{-D}\phi_i(x) = (1+\epsilon)^{-D-1}\partial_\mu\phi_i(x),$$

we have $\delta\partial_\mu\phi_i = -(D+1)\epsilon\partial_\mu\phi_i$. [This may also be seen from the general equation $\delta\partial_\mu\phi_i = (\delta x^\nu)\partial_\mu\partial_\nu\phi_i + \partial_\mu\mathfrak{d}\phi_i = \partial_\mu\delta\phi_i - (\partial_\mu\delta x^\nu)\partial_\nu\phi_i$, which in our case is $\partial_\mu(-D\epsilon)\phi_i - (\partial_\mu\epsilon x^\nu)\partial_\nu\phi_i = -(D+1)\epsilon\partial_\mu\phi_i$.]

In this case the first term in (1) does give a contribution,

$$\partial_\mu\delta x^\mu = \epsilon\delta^\mu_\mu = 4\epsilon,$$

so

$$\begin{aligned}\delta\mathcal{L} &= \epsilon\left(4\mathcal{L} + Dm^2\phi^2 - (D+1)(\partial_\mu\phi_i)\partial^\mu\phi_i\right)\\ &= \epsilon\left[(1-D)(\partial_\mu\phi_i)\partial^\mu\phi_i + (D-2)m^2\phi^2\right].\end{aligned}$$

For this to vanish, the first term requires $D = 1$, the canonical dimension for a scalar field (in four dimensional space-time), but then the second term vanishes only if $m^2 = 0$. That is, dilation is a symmetry only for the **massless** scalar field. This is not surprising, as a fixed mass sets a scale of mass, energy, and length, the latter two because $c = 1$ and $\hbar = 1$ respectively.

With $m = 0$, the dilatation current is

$$J^\mu = (\partial^\mu\phi_i)\left((\partial_\nu\phi_i)x^\nu + \phi_i\right) - \mathcal{L}x^\mu = T^\mu{}_\nu x^\nu + \frac{1}{2}\partial^\mu\phi^2.$$

The dilation charge is therefore

$$Q = \int d^3x J^0 = \int d^3x (T^0{}_\nu x^\nu + \phi\dot{\phi}).$$