Physics 615 Fall, 2007
Homework Solutions #1

1) From the Euler-Lagrange equations for $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{2}m^2\phi_i\phi_i$,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\partial_\mu\phi_i} - \frac{\partial\mathcal{L}}{\partial\phi_i} = 0 \Rightarrow \partial_\mu\partial^\mu\phi_i + m^2\phi_i = 0.$$

Thus both components obey $(\partial_\mu\partial^\mu + m^2)\,\phi_i = 0$, and therefore so do the linear combinations ϕ and $\phi^\dagger$, where

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}.$$

In terms of ϕ and $\phi^\dagger$,

$$\phi^\dagger\phi = \frac{1}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \frac{1}{2}(\phi_1^2 + \phi_2^2) = \frac{1}{2}\phi_i\phi_i,$$

and

$$\partial_\mu\phi^\dagger\partial^\mu\phi = \frac{1}{2}(\partial_\mu\phi_1 - i\partial_\mu\phi_2)(\partial^\mu\phi_1 + i\partial^\mu\phi_2) = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i,$$

so

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi.$$

If we pretended that ϕ and $\phi^\dagger$ are independent individual degrees of freedom, we would write

$$\begin{aligned}
\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} - \frac{\partial\mathcal{L}}{\partial\phi} &= \left(\partial_\mu\partial^\mu\phi^\dagger + m^2\phi^\dagger\right) = 0,\\
\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\dagger} - \frac{\partial\mathcal{L}}{\partial\phi^\dagger} &= \left(\partial_\mu\partial^\mu\phi + m^2\phi\right) = 0,
\end{aligned}$$

which are the same equations, of course.

For the Hamiltonian density

$$\mathcal{H} = \sum_{j=1}^{2}\pi_i\dot\phi_i - \mathcal{L} = \frac{1}{2}\sum_{j=1}^{2}(\dot\phi_i)^2 + \frac{1}{2}\sum_{j=1}^{2}(\vec\nabla\phi_i)^2 + \frac{m^2}{2}\sum_{j=1}^{2}\phi_i^2,$$

as

$$\pi_i = \frac{\delta L}{\delta\dot\phi_i} = \frac{\partial\mathcal{L}}{\partial\dot\phi_i}.$$

This can be reexpressed in terms of the complex field:

$$\mathcal{H} = \dot\phi^\dagger\dot\phi + \vec\nabla\phi^\dagger\cdot\vec\nabla\phi + m^2\phi^\dagger\phi.$$

If we treat ϕ and $\phi^\dagger$ as independent, we would define

$$\begin{aligned}
\pi &= \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi^\dagger\\
\pi^\dagger &= \frac{\partial\mathcal{L}}{\partial\dot\phi^\dagger} = \dot\phi.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{H} &= \pi^\dagger\dot\phi^\dagger + \pi\dot\phi - \mathcal{L} = \dot\phi\dot\phi^\dagger + \dot\phi^\dagger\dot\phi - \left(\dot\phi^\dagger\dot\phi - \vec\nabla\phi^\dagger\cdot\vec\nabla\phi - m^2\phi^\dagger\phi\right)\\
&= \dot\phi\dot\phi^\dagger + \vec\nabla\phi^\dagger\cdot\vec\nabla\phi + m^2\phi^\dagger\phi,
\end{aligned}$$

exactly the same as when "done right".

2) With

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

we have

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu\right)$$

Then using the covariant forms of variations, $\tilde\delta$, we have

$$\frac{\partial\mathcal{L}}{\partial A^\mu} = 0, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu},$$

so the equations of motion are

$$\partial_\nu\partial^\mu A^\nu - \partial_\nu\partial^\nu A^\mu = \partial_\nu F^{\mu\nu} = 0.$$

With $E^j = F^{j0}$ and $\epsilon_{ijk}B^k = -F^{ij}$ (and therefore $B^k = -\frac{1}{2}\epsilon_{kij}F^{ij}$), this gives, for $\mu = 0$

$$\partial_j F^{0j} = -\vec\nabla\cdot\vec E = 0,$$

while for $\mu = j$ we have

$$\partial_0 F^{j0} + \partial_i F^{ji} = \dot{E}^j + \partial_i \epsilon_{ijk} B^k = \dot{\vec{E}} - \vec{\nabla} \times \vec{B} = 0.$$

The first of these is Gauss's law in vacuum, and the second is Ampère's law, including the displacement current. What about the other two laws? The magnetic version of Gauss says

$$\vec{\nabla} \cdot \vec{B} = \partial_i B^i = \frac{1}{2}\epsilon_{ijk}\partial_i F^{jk} = \frac{1}{2}\epsilon_{ijk}\partial_i(\partial_j A_k - \partial_k A_j) = 0$$

because $\partial_i \partial_j$ is symmetric under $i \leftrightarrow j$ while the ϵ is antisymmetric. And Faraday's law

$$\begin{aligned}\left(\vec{\nabla} \times \vec{E} + \dot{\vec{B}}\right)_j &= \epsilon_{jk\ell}\partial_k E_\ell + \partial_0 B_j = \epsilon_{jk\ell}\partial_k F^{\ell 0} - \frac{1}{2}\partial_0 \epsilon_{jk\ell} F^{k\ell} \\ &= \epsilon_{jk\ell}\left[\partial_k\left(\partial^\ell A^0 - \partial^0 A^\ell\right) - \partial_0 \partial^k A^\ell\right] = 0.\end{aligned}$$

Thus the magnetic version of Gauss' law and Faraday's law are consequences of the treatment of A^μ as the degrees of freedom, while Ampère's law and the electric version of Gauss' law, which are modified in the presence of charges and currents, are consequences of the equations of motion. To introduce charges and currents, we will need to add an interaction term to the Lagrangian,

To derive the canonical momenta

$$\Pi_\mu := \frac{\delta L}{\delta \dot{A}^\mu} = \partial_\mu A^0 - \partial_0 A_\mu = F_{\mu 0},$$

so

$$\left(\vec{\Pi}\right)_j = \Pi^j = F^{j0} = E_j, \qquad \text{but } \Pi_0 = F^{00} = 0.$$

One of the momenta is thus **identically** 0, and we cannot solve for the four $\dot{A}^\mu$ in terms of the three independent Π^μ. We can write

$$\dot{A}^i = \partial^i A^0 - F^{i0} = \partial^i A^0 - \Pi^i$$

but we have no equation for $\dot{A}^0$.

3) As before,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu},$$

but now

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = -j_\mu,$$

so the equations of motion are

$$\begin{aligned} 0 &= \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} - \frac{\partial \mathcal{L}}{\partial A^\mu} \\ &= \partial_\nu F^{\mu\nu} + j^\mu \end{aligned}$$

With $E^k = F^{k0}$ and $\epsilon_{ik\ell} B^\ell = -F^{ik}$ (and therefore $B^k = -\frac{1}{2}\epsilon_{ki\ell}F^{i\ell}$), this gives, for $\mu = 0$

$$\partial_k F^{0k} = -\vec{\nabla} \cdot \vec{E} = -j^0,$$

while for $\mu = k$ we have

$$\partial_0 F^{k0} + \partial_i F^{ki} = \dot{E}^k + \partial_i \epsilon_{ik\ell} B^\ell = \dot{\vec{E}} - \vec{\nabla} \times \vec{B} = -j^k.$$

The first of these is Gauss's law in vacuum, and the second is Ampère's law, including the displacement current.

What about the other two laws, $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = 0$? They came directly from the expression for $F^{\mu\nu}$ in terms of A^ρ, so are independent of the equations of motion, and unchanged by the source term.

As

$$\partial_\nu F^{\mu\nu} = j^\mu,$$

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = \frac{1}{2}\left[\partial_\mu, \partial_\nu\right] F^{\mu\nu} = 0,$$

because $F^{\mu\nu} = -F^{\nu\mu}$, and we are summing over dummy indices. Thus unless $\partial_\mu j^\mu = 0$ we have an inconsistence with the equations of motion. This is the requirement of **current conservation**.