Physics 615 Nov. 1, 2007
Homework Solutions #8

1 [10 pts] Do problem 5.2 from Peskin and Schroeder, which is to calculate Bhabha scattering, the e^+e^- differential elastic scattering, unpolarized, to lowest order in QED. You may assume $E_{\rm cm} \gg m_e$, so set m_e to zero, except that in discussing the divergence when $\theta \to 0$, please explain whether the electron mass removes the divergence.

Solution 1 There are two contributions to the amplitude we need to calculate,

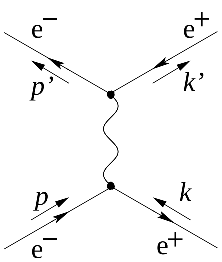
$$(2\pi)^4\delta^4(p+k-p'-k')i\mathcal{M} = \int \frac{d^4x}{(2\pi)^4}\int\frac{d^4y}{(2\pi)^4}\langle p'k'|\left(-ie\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x)\right)\left(-ie\bar\psi(y)\gamma^\nu A_\nu(y)\psi(y)\right)|pk\rangle\,,$$

depending on whether the H_I that annihilates the incoming electron also annihilates the incoming positron, or whether it creates the outgoing electron. The first contraction involves

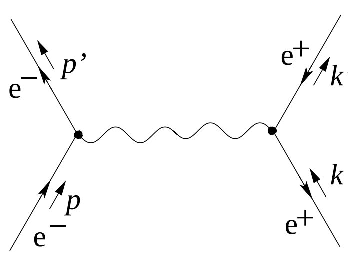
$$\langle 0|\, b_{k'} a_{p'} \bar\psi\gamma^\mu\psi\bar\psi\gamma_\mu\psi a_p^\dagger b_k^\dagger\,|0\rangle \quad \sim +\bar u(p')\gamma^\mu v(k')\bar v(k)\gamma_\mu u(p) \qquad \text{Fig (a)}$$

and the second is

$$\langle 0|\, b_{k'} a_{p'} \bar\psi\gamma^\mu\psi\bar\psi\gamma_\mu\psi a_p^\dagger b_k^\dagger\,|0\rangle \quad \sim -\bar v(k)\gamma^\mu v(k')\bar u(p')\gamma_\mu u(p) \qquad \text{Fig (b)}$$



(a)



(b)

Thus the scattering amplitude is

$$\begin{aligned}\mathcal{M} &= \bar u(p')(-ie\gamma_\mu)v(k')\,\bar v(k)(-ie\gamma_\nu)u(p)\frac{-ig^{\mu\nu}}{s+i\epsilon} \\ &\quad -\bar v(k)(-ie\gamma_\mu)v(k')\,\bar u(p')(-ie\gamma_\nu)u(p)\frac{-ig^{\mu\nu}}{t+i\epsilon} \\ &= ie^2\frac{\bar u(p')\gamma^\mu v(k')\,\bar v(k)\gamma_\mu u(p)}{s+i\epsilon} - ie^2\frac{\bar v(k)\gamma^\mu v(k')\,\bar u(p')\gamma_\mu u(p)}{t+i\epsilon}.\end{aligned}$$

Working in the center of mass, summing over final spins and averaging over initial spins, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac14\sum_{rr'ss'}\frac{1}{64\pi^2E_{\rm cm}}|\mathcal{M}(ps+kr\to p's'+k'r')|^2 = \frac{e^4}{256\pi^2 s}\sum_{rr'ss'} \\ &\quad \left(\frac{\bar u^{s'}(p')\gamma^\mu v^{r'}(k')\,\bar v^r(k)\gamma_\mu u^s(p)}{s+i\epsilon} - \frac{\bar v^r(k)\gamma^\mu v^{r'}(k')\,\bar u^{s'}(p')\gamma_\mu u^s(p)}{t+i\epsilon}\right) \\ &\quad \times\left(\frac{\bar v^{r'}(k')\gamma^\nu u^{s'}(p')\,\bar u^s(p)\gamma_\nu v^r(k)}{s+i\epsilon} - \frac{\bar v^{r'}(k')\gamma^\nu v^r(k)\,\bar u^s(p)\gamma_\nu u^{s'}(p')}{t+i\epsilon}\right) \\ &= \frac{e^4}{256\pi^2 s}\sum_{rr'ss'}\Bigg(\\ &\quad \frac{\bar u^{s'}(p')\gamma^\mu v^{r'}(k')\,\bar v^{r'}(k')\gamma^\nu u^{s'}(p')\;\; \bar v^r(k)\gamma_\mu u^s(p)\,\bar u^s(p)\gamma_\nu v^r(k)}{s^2} \\ &\quad -\frac{\bar u^{s'}(p')\gamma^\mu v^{r'}(k')\,\bar v^{r'}(k')\gamma^\nu v^r(k)\;\bar v^r(k)\gamma_\mu u^s(p)\,\bar u^s(p)\gamma_\nu u^{s'}(p')}{st} \\ &\quad -\frac{\bar v^r(k)\gamma^\mu v^{r'}(k')\,\bar v^{r'}(k')\gamma^\nu u^{s'}(p')\;\bar u^{s'}(p')\gamma_\mu u^s(p)\,\bar u^s(p)\gamma_\nu v^r(k)}{st} \\ &\quad +\frac{\bar v^r(k)\gamma^\mu v^{r'}(k')\,\bar v^{r'}(k')\gamma^\nu v^r(k)\;\bar u^{s'}(p')\gamma_\mu u^s(p)\,\bar u^s(p)\gamma_\nu u^{s'}(p')}{t^2}\Bigg)\end{aligned}$$

Let us pause here for some simplifications. First, $\alpha := e^2/4\pi$. Then note $s=(p+k)^2 = 2p\cdot k = (p'+k')^2 = 2p'\cdot k'$, $t=(p'-p)^2 = -2p\cdot p' = (k'-k)^2 = -2k\cdot k'$, and $u=-s-t=(p'-k)^2 = -2p'\cdot k = (p-k')^2 = -2p\cdot k'$, where we have used $p^2 = p'^2 = k^2 = k'^2 = m_e^2 \approx 0$, and $s+t+u = 4m_e^2\approx 0$. Now

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{16s}\Big(\frac{1}{s^2}\,\mathrm{Tr}\,(\not{p}'\gamma^\mu\not{k}'\gamma^\nu)\,\mathrm{Tr}\,(\not{k}\gamma_\mu\not{p}\gamma_\nu) - \frac{1}{st}\,\mathrm{Tr}\,(\not{p}'\gamma^\mu\not{k}'\gamma^\nu\not{k}\gamma_\mu\not{p}\gamma_\nu) \\ &\quad -\frac{1}{st}\,\mathrm{Tr}\,(\not{k}\gamma^\mu\not{k}'\gamma^\nu\not{p}'\gamma_\mu\not{p}\gamma_\nu) + \frac{1}{t^2}\,\mathrm{Tr}\,(\not{k}\gamma^\mu\not{k}'\gamma^\nu)\,\mathrm{Tr}\,(\not{p}'\gamma_\mu\not{p}\gamma_\nu)\Big)\end{aligned}$$

The traces are

$$\begin{aligned}
\mathrm{Tr}\,(\not{k}\gamma_\mu \not{p}\gamma_\nu) &= 4(k_\mu p_\nu + k_\nu p_\mu - k\cdot p g_{\mu\nu}) \\
\mathrm{Tr}\,(\not{p}'\gamma^\mu \not{k}'\gamma^\nu \not{k}\gamma_\mu \not{p}\gamma_\nu) &= -2\,\mathrm{Tr}\,(\not{p}'\gamma^\mu \not{k}'\not{p}\gamma_\mu \not{k}) = -8p\cdot k'\,\mathrm{Tr}\,(\not{p}'\not{k}) \\
&= -32p\cdot k'\, p'\cdot k, \\
\mathrm{Tr}\,(\not{k}\gamma^\mu \not{k}'\gamma^\nu \not{p}'\gamma_\mu \not{p}\gamma_\nu) &= -2\,\mathrm{Tr}\,(\not{k}\gamma^\mu \not{k}'\not{p}\gamma_\mu \not{p}') = -8k'\cdot p\,\mathrm{Tr}\,(\not{k}\not{p}') \\
&= -32k'\cdot p\, k\cdot p'.
\end{aligned}$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{s}\Big((s^{-2}\,(p'^\mu k'^\nu + p'^\nu k'^\mu - p'\cdot k' g^{\mu\nu})\,(k_\mu p_\nu + k_\nu p_\mu - p\cdot k g_{\mu\nu}) \\
&\quad +4s^{-1}t^{-1}k'\cdot p\, k\cdot p' \\
&\quad +t^{-2}\,(k^\mu k'^\nu + k^\nu k'^\nu - k\cdot k' g^{\mu\nu})\,\big(p'_\mu p_\nu + p'_\nu p_\mu - p'\cdot p g_{\mu\nu}\big)\Big) \\
&= \frac{\alpha^2}{s}\Big(s^{-2}\,(2p'\cdot k\, k'\cdot p + 2k\cdot k'\, p\cdot p') \\
&\quad +t^{-2}\,(2k\cdot p'\, k'\cdot p + 2k\cdot p\, k'\cdot p') + 4s^{-1}t^{-1}k\cdot p'\, k'\cdot p\Big)
\end{aligned}$$

Once again we use s, t and u, with

$$p\cdot k = p'\cdot k' = s/2, \quad p\cdot k' = p'\cdot k = -u/2 = \frac{1}{2}(s+t), \quad k\cdot k' = p\cdot p' = -t/2.$$

so

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2s}\Big(\frac{u^2+t^2}{s^2} + \frac{u^2+s^2}{t^2} + 2\frac{u^2}{st}\Big) \\
&= \frac{\alpha^2}{2s}\Big(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \frac{t^2}{s^2} + \frac{s^2}{t^2}\Big).
\end{aligned}$$

Integrating $\int_0^{2\pi} d\phi$ gives

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\Big(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \frac{t^2}{s^2} + \frac{s^2}{t^2}\Big).$$

In terms of scattering angles,

$$t = -\frac{s}{2}(1-\cos\theta) = -s\sin^2(\theta/2), \qquad u = -s + \frac{s}{2}(1-\cos\theta) = -s\cos^2(\theta/2),$$

so

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left(\frac{\cos^8(\theta/2)}{\sin^4(\theta/2)} + \sin^4(\theta/2) + \frac{1}{\sin^4(\theta/2)}\right).$$

or

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2.$$

The divergence as $\theta \to 0$ comes from $t = -2\vec{p}^2(1-\cos\theta) \to 0$, which is correct even if we don't ignore the masses, and the fact that $|\mathcal{M}|^2$ has a term which goes like t^{-2} due to the masslessness of the exchanged photon. This is nothing new — the Rutherford crosssection also has a $\sin^4(\theta/2)$ in the denominator.