Physics 615 Oct. 18, 2007
Homework Solutions #6

1 [15 pts] Do problem 3.5 from Peskin and Schroeder. This is the simplest presentation of a model with supersymmetry. Note that the F field does not correspond to particles, as its field equation is not an equation of motion, so to express the particle content of this action F needs to be solved for and the expression substituted back into the Lagrangian.

Solution 1 (a) We are going to consider the theory with Lagrangian density

$$\mathcal{L}_0 = (\partial_\mu \phi^*)\partial^\mu \phi + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + F^* F, \tag{1}$$

where ϕ is a complex scalar field, χ a two component left handed Weyl spinor, and F a complex scalar which will turn out not to describe any particles. We are going to consider a symmetry with an infinitesimal global Grassmann parameter which transforms like a left handed Weyl spinor producing the infinitesimal changes in the fields: Under the changes

$$\begin{aligned} \delta\phi &= -i\epsilon^T \sigma^2 \chi, & (2)\\ \delta\chi &= \epsilon F + \sigma^\mu \partial_\mu \phi\, \sigma^2 \epsilon^*, & (3)\\ \delta F &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi. & (4) \end{aligned}$$

This produces the change in $\mathcal{L}_0$ given by

$$\begin{aligned} \delta\mathcal{L}_0 &= (-i\epsilon^T\sigma^2\partial_\mu\chi)^*\partial^\mu\phi + (\partial_\mu\phi^*)\partial^\mu(-i\epsilon^T\sigma^2\chi) \\ &\quad +(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*)^\dagger i\bar{\sigma}^\nu\partial_\nu\chi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu(\epsilon F + \sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) \\ &\quad +(-i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi)^* F - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi \\ &= i(\partial_\mu\chi^\dagger)\sigma^2\epsilon^*\partial^\mu\phi - i(\partial_\mu\phi^*)\epsilon^T\sigma^2\partial^\mu\chi \\ &\quad +iF^*\epsilon^\dagger\bar{\sigma}^\nu\partial_\nu\chi + i\epsilon^T\sigma^2\sigma^\mu\bar{\sigma}^\nu(\partial_\nu\chi)\partial_\mu\phi^* + \chi^\dagger i\bar{\sigma}^\mu\epsilon\partial_\mu F \\ &\quad +i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^* + i(\partial_\mu\chi^\dagger)\bar{\sigma}^\mu\epsilon F - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi \\ &= i\partial_\mu(\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi) - i\chi^\dagger\sigma^2\epsilon^*\partial_\mu\partial^\mu\phi \\ &\quad -i\partial^\mu((\partial_\mu\phi^*)\epsilon^T\sigma^2\chi) + i(\partial^\mu\partial_\mu\phi^*)\epsilon^T\sigma^2\chi \end{aligned}$$

$$
\begin{aligned}
& +i\partial_\nu(\epsilon^T\sigma^2\sigma^\mu\bar\sigma^\nu\chi\partial_\mu\phi^*) - i\epsilon^T\sigma^2\sigma^\mu\bar\sigma^\nu\chi\partial_\nu\partial_\mu\phi^* \\
& +i\partial_\mu(\chi^\dagger\bar\sigma^\mu\epsilon F) - i(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F \\
& +i\chi^\dagger\sigma^2\epsilon^*(\partial_\mu\partial^\mu\phi) + i(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F \\
= \; & i\partial_\mu\Big(\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi - (\partial_\mu\phi^*)\epsilon^T\sigma^2\chi \\
& \qquad +i\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu\chi\partial_\nu\phi^* + \chi^\dagger\bar\sigma^\mu\epsilon F\Big) \\
& -i\chi^\dagger\sigma^2\epsilon^*\partial_\mu\partial^\mu\phi + i(\partial^\mu\partial_\mu\phi^*)\epsilon^T\sigma^2\chi \\
& -i\epsilon^T\sigma^2\chi\partial^\mu\partial_\mu\phi^* + i\chi^\dagger\sigma^2\epsilon^*(\partial_\mu\partial^\mu\phi) \\
= \; & i\partial_\mu\Big(\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi - (\partial_\mu\phi^*)\epsilon^T\sigma^2\chi + i\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu\chi\partial_\nu\phi^* + \chi^\dagger\bar\sigma^\mu\epsilon F\Big)
\end{aligned}
$$

(b) For

$$\Delta\mathcal{L} = [m\phi F + \frac{im}{2}\chi^T\sigma^2\chi] + \text{hermitean conjugate},$$

$$
\begin{aligned}
\delta(m\phi F) &= -im\epsilon^T\sigma^2\chi F - im\phi\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi, \\
\delta(\chi^T\sigma^2\chi) &= (\delta\chi)^T\sigma^2\chi + \chi^T\sigma^2\delta\chi = 2\chi^T\sigma^2\delta\chi, \quad \text{so} \\
\delta(\frac{im}{2}\chi^T\sigma^2\chi) &= im\chi^T\sigma^2\epsilon F + im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\,\sigma^2\epsilon^* \\
&= im\epsilon^T\sigma^2\chi F - im\epsilon^\dagger\bar\sigma^\mu\chi\partial_\mu\phi
\end{aligned}
$$

so

$$\delta\Delta\mathcal{L} = -im\partial_\mu(\epsilon^\dagger\bar\sigma^\mu\chi\phi) \sim 0.$$

Now let us find the "equation of motion" for F which follows from the full Lagrangian $\mathcal{L} = \mathcal{L}_0 + \Delta\mathcal{L}$. As there are no derivatives of F in this Lagrangian, the field equation is $\delta\mathcal{L}/\delta F = 0 = F^* + m\phi$, so $F = -m\phi^*, F^* = -m\phi$, which we see are **not equations of motion**, but rather constraint equations. Eliminating F and F^* by substitution into the lagrangian, we find

$$\mathcal{L} = (\partial_\mu\phi^*)\partial^\mu\phi + \chi^\dagger i\bar\sigma^\mu\partial_\mu\chi - m^2\phi^*\phi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*).$$

We saw in the last problem that the $\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*$ term gives χ a mass m, while homework #1 problem #1 showed us $-m^2\phi^*\phi$ gives the ϕ particles a mass m.

(c) The variation of $G[\phi]$, a function of several ϕ_i but not explicitly of ϕ_i^* or the other fields, under an infinitesimal transformation is just

$$\delta G[\phi] = \frac{\partial G[\phi]}{\partial \phi_i}\delta\phi_i.$$

Applying this to $G = \partial W[\phi]/\partial\phi_i$ and $G' = \partial^2 W[\phi]/\partial\phi_i\partial\phi_j$, in

$$\Delta\mathcal{L} = F_i \frac{\partial W[\phi]}{\partial\phi_i} + \frac{i}{2}\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + \text{ hermitean conjugate,}$$

we have

$$\begin{aligned}\delta\Delta\mathcal{L} &= \delta F_i \frac{\partial W[\phi]}{\partial\phi_i} + F_i\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\delta\phi_j + \frac{i}{2}\frac{\partial^3 W[\phi]}{\partial\phi_i\partial\phi_j\partial\phi_k}(\delta\phi_k)\chi_i^T\sigma^2\chi_j \\ &\quad + i\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\delta\chi_j + \text{ hermitean conjugate.}\end{aligned}$$

Under the single supersymmetry transformation

$$\begin{aligned}\delta\phi_j &= -i\epsilon^T\sigma^2\chi_j, \\ \delta\chi_j &= \epsilon F_j + \sigma^\mu\partial_\mu\phi_j\,\sigma^2\epsilon^*, \\ \delta F_j &= -i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_j.\end{aligned}$$

we have

$$\begin{aligned}\delta\Delta\mathcal{L} &= -i\frac{\partial W[\phi]}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_j - iF_i\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\epsilon^T\sigma^2\chi_j \\ &\quad + \frac{1}{2}\frac{\partial^3 W[\phi]}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j\epsilon^T\sigma^2\chi_k \\ &\quad + i\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2(\epsilon F_j + \sigma^\mu\partial_\mu\phi_j\,\sigma^2\epsilon^*) + \text{ hermitean conjugate}\end{aligned}$$

The second term on the first line cancels the ϵF term in the last line, and the $\chi_i^T\sigma^2\sigma^\mu\partial_\mu\phi_j\,\sigma^2\epsilon^*$ can be simplified to $-\epsilon^\dagger\bar\sigma^\mu\chi_i\partial_\mu\phi_j$ but how can we possibly cancel the first term, the only one involving only a single derivative of the arbitrary function W? Remember that we need to cancel it only up to a total derivative, and

$$-i\frac{\partial W[\phi]}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_i = -i\partial_\mu\Big(\frac{\partial W[\phi]}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\chi_i\Big) + i\epsilon^\dagger\bar\sigma^\mu\chi_i\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\partial_\mu\phi_j,$$

and the last term cancels the one we just simplified. This leaves us with only the

$$\frac{\partial^3 W[\phi]}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j\epsilon^T\sigma^2\chi_k$$

term. The first factor is totally symmetric under permutations of (i, j, k), while the remaining factor is proportional to $\chi_i^r\chi_j^s\chi_k^t$, where r, s, t are the spin indices which can only take on the two values 1 and 2. But this Grassman product is totally antisymmetric under any combined permution of (ijk) with the same permutation of (rst), and as it is multiplied by something unchanged by the former, only the part totally antisymmetric under permutations of (rst) will survive, and there cannot be any such as two of r, s, t must be equal. Thus we have shown the variation of the full Lagrangian is invariant up to a total derivative under the supersymmetry.

For $W = g\phi^3/3$ with only one set of fields,

$$\mathcal{L} = \mathcal{L}_0 + gF\phi^2 + ig\phi\chi^T\sigma^2\chi + gF^*\phi^{2\,*} + ig\phi^*\chi^\dagger\sigma^2\chi^*.$$

The field equations for the auxiliary fields give $F = -g\phi^{2\,*}$ and $F^* = -g\phi^2$, so the remaining equations are

$$\begin{aligned} 0 &= \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*} - \frac{\delta\mathcal{L}}{\delta\phi^*} = \partial_\mu\partial^\mu\phi - 2gF^*\phi^* + ig\chi^\dagger\sigma^2\chi^* \\ &= \partial_\mu\partial^\mu\phi + 2g^2\phi^2\phi^* + ig\chi^\dagger\sigma^2\chi^* \\ 0 &= \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\chi^\dagger} - \frac{\delta\mathcal{L}}{\delta\chi^\dagger} = i\bar\sigma^\mu\partial_\mu\chi + 2ig\phi^*\sigma^2\chi^*. \end{aligned}$$

Thus we have a theory with Weyl fermions coupled with a Yukawa coupling to scalar fields, but the charges are not obvious. While $\mathcal{L}_0$ is invariant under separate phase changes for ϕ, F and χ, the interaction $\Delta\mathcal{L}$, with $W \propto \phi^3$, requires the charge of F to be -2 times the charge of ϕ, and that of ϕ to be -2 times that of χ. So we do have charge conservation with $q_F = 4q_\chi, q_\phi = -2q_\chi$.