Physics 615 Sept. 27, 2007
Homework Solution #3

1) In lecture we defined the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma},$$

built of the generators of the Poincaré group. Show that

(a) $[W^\mu, P_\nu] = 0$,

and that its square is a Casimir operator of the Poincaré group, that is

(b) $[W^\mu W_\mu, P_\nu] = 0$,

(c) $[W^\mu W_\mu, L_{\alpha\beta}] = 0$.

Also show, if you haven't already, that P^2 is also a Casimir operator of the group.

It may be useful to examine whether your evaluation of $[L_{\alpha\beta}, W^\mu]$ is what you would expect for a general vector, $[L_{\alpha\beta}, V^\mu]$, and also to show that the vector properties implied by the indices do transform correctly under commutator with $L_{\alpha\beta}$. That is, $V^\mu F_\mu$ should commute, and the c-numbers $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$, which of course do commute with all operators, **should** commute with Lorentz transformations despite having Lorentz indices.

Solution 1 We will need the commutators of L's and P's with each other,

$$\begin{aligned}
[L_{\alpha\beta}, L_{\gamma\zeta}] &= ig_{\alpha\gamma}\, L_{\beta\zeta} - ig_{\beta\gamma}\, L_{\alpha\zeta} - ig_{\alpha\zeta}\, L_{\beta\gamma} + ig_{\zeta\beta}\, L_{\alpha\gamma} && (1)\\
[L_{\alpha\beta}, P_\nu] &= -ig_{\beta\nu} P_\alpha + ig_{\alpha\nu} P_\beta && (2)\\
[P^\mu, P^\nu] &= 0 && (3)
\end{aligned}$$

As $W^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha L_{\beta\gamma}$,

$$[W^\mu, P_\nu] = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha\, [L_{\beta\gamma}, P_\nu] = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha \left(-ig_{\gamma\nu} P_\beta + ig_{\beta\nu} P_\gamma\right) = 0 \qquad (4)$$

so, of course, $[W^\mu W_\mu, P_\nu] = 0$.

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\, [P_\alpha, L_{\nu\rho}]\, L_{\beta\gamma} + \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha\, [L_{\beta\gamma}, L_{\nu\rho}]\\
&= \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} \left(ig_{\rho\alpha} P_\nu - ig_{\nu\alpha} P_\rho\right) L_{\beta\gamma}\\
&\quad + \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha \left(ig_{\beta\nu}\, L_{\gamma\rho} - ig_{\gamma\nu}\, L_{\beta\rho} - ig_{\beta\rho}\, L_{\gamma\nu} + ig_{\rho\gamma}\, L_{\beta\nu}\right)
\end{aligned} \qquad (5)$$

Do we believe this? A vector should transform as P does, so we would expect

$$[W^\mu, L_{\nu\rho}] = i\delta^\mu_\rho W_\nu - i\delta^\mu_\nu W_\rho. \qquad (6)$$

There are two ways we can verify that, in fact, (6) is correct. The first is to notice that because of the contraction with the epsilon, the α, β, γ indices in the parentheses in (5) may be permuted, changing the sign if necessary, so we can replace $g_{\beta\nu} P_\alpha L_{\gamma\rho}$ and $-g_{\gamma\nu} P_\alpha L_{\beta\rho}$ with $g_{\alpha\nu} P_\gamma L_{\beta\rho}$ and $-g_{\alpha\nu} P_\beta L_{\gamma\rho}$ and also replace $-g_{\beta\rho} P_\alpha L_{\gamma\nu}$ and $g_{\rho\gamma} P_\alpha\, L_{\beta\nu}$ with $g_{\alpha\rho} P_\beta L_{\gamma\nu}$ and $-g_{\alpha\rho} P_\gamma L_{\beta\nu}$. Thus

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{i}{2}\epsilon^{\mu\alpha\beta\gamma} \Big(g_{\alpha\rho} \left\{ P_\nu L_{\beta\gamma} + P_\beta L_{\gamma\nu} + P_\gamma L_{\nu\beta} \right\}\\
&\qquad - g_{\alpha\nu} \left\{ P_\rho L_{\beta\gamma} + P_\beta L_{\gamma\rho} + P_\gamma L_{\rho\beta} \right\} \Big)
\end{aligned} \qquad (7)$$

The expression in each of the $\{\}$'s is totally antisymmetric in the three indices, and can therefore be written in terms of W, for

$$\epsilon_{\rho\beta\gamma\mu} W^\mu = \frac{1}{2}\epsilon_{\rho\beta\gamma\mu}\epsilon^{\mu\alpha\sigma\tau} P_\alpha L_{\sigma\tau} = P_\rho L_{\gamma\beta} + P_\beta L_{\rho\gamma} + P_\gamma L_{\beta\rho}.$$

Thus we see that

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{i}{2}\epsilon^{\mu\alpha\beta\gamma} \Big(g_{\alpha\rho}\epsilon_{\beta\nu\gamma\tau} W^\tau - g_{\alpha\nu}\epsilon_{\beta\rho\gamma\tau} W^\tau \Big)\\
&= \frac{i}{2} \Big(g_{\alpha\rho}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\tau\nu\beta\gamma} W^\tau + g_{\alpha\nu}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\rho\tau\beta\gamma} W^\tau \Big)\\
&= i \Big(g_{\alpha\rho} (\delta^\mu_\tau \delta^\alpha_\nu - \delta^\alpha_\tau \delta^\mu_\nu) W^\tau + g_{\alpha\nu} (\delta^\mu_\rho \delta^\alpha_\tau - \delta^\alpha_\rho \delta^\mu_\tau) W^\tau \Big)\\
&= ig_{\nu\rho} W^\mu - i\delta^\mu_\nu W_\rho + i\delta^\mu_\rho W_\nu - ig_{\rho\nu} W^\mu\\
&= i\delta^\mu_\rho W_\nu - i\delta^\mu_\nu W_\rho,
\end{aligned} \qquad (8)$$

just what it should be!

The second way to find that W^μ transforms by (6) the way it should is to prove our suspicions, that quantities made with vector indices transform the way we think under commutator with $L_{\alpha\beta}$, that is, any vector V_ν obeys

$$[L_{\alpha\beta}, V_\nu] = -ig_{\beta\nu} V_\alpha + ig_{\alpha\nu} V_\beta$$

just as P does in (2). A tensor with two indices should have each one transform that way,

$$[L_{\alpha\beta}, T_{\gamma\zeta}] \;=\; -ig_{\beta\gamma} T_{\alpha\zeta} + ig_{\alpha\gamma} T_{\beta\zeta} - ig_{\beta\zeta} T_{\gamma\alpha} + ig_{\alpha\zeta} T_{\gamma\beta}$$

If the tensor is antisymmetric, like L itself,

$$[L_{\alpha\beta}, T_{\gamma\zeta}] = ig_{\alpha\gamma}\, T_{\beta\zeta} - ig_{\beta\gamma}\, T_{\alpha\zeta} - ig_{\alpha\zeta}\, T_{\beta\gamma} + ig_{\zeta\beta}\, T_{\alpha\gamma},$$

which is Eq. (1) if T is L. So the Lorentz tensors do transform correctly.

Does $g_{\mu\nu}$ transform correctly? As it is a c-number, it commutes with $L_{\alpha\beta}$, but if it transforms as a tensor,

$$[L_{\alpha\beta}, g_{\mu\nu}] = -ig_{\beta\mu} g_{\alpha\nu} + ig_{\alpha\mu} g_{\beta\nu} - ig_{\beta\nu} g_{\mu\alpha} + ig_{\alpha\nu} g_{\mu\beta} = 0$$

as g is symmetric. So we see that not changing is just what it is supposed to do.

Do contractions behave correctly? If V_ν and W_ν transform as P_ν, then

$$\begin{aligned}[L_{\alpha\beta}, V_\nu W^\nu] &= -ig_{\beta\nu} V_\alpha W^\nu + ig_{\alpha\nu} V_\beta W^\nu - i\delta_\beta{}^\nu V_\nu W_\alpha + i\delta_\alpha{}^\nu V_\nu W_\beta \\ &= -iV_\alpha W_\beta + iV_\beta W_\alpha - iV_\beta W_\alpha + iV_\alpha W_\beta = 0\end{aligned}$$

so contracted indices don't contribute to the commutator. This means any compound object we make of components which transform properly by commutation with $L_{\alpha\beta}$ will also do so.

In particular, that means the Minkowski square of any vector which transforms properly under Lorentz transformations will commute with the Lorentz generators, and will be a Casimir operator of the Lorentz group, Thus P^2, which also commutes with the Poincaré generators P_ν, is a Casimir operator of the Poincaré group.

Now for constructing W^μ we also need the ϵ tensor. For $\epsilon_{\mu\nu\rho\sigma}$

$$\begin{aligned}[L_{\alpha\beta}, \epsilon_{\mu\nu\rho\sigma}] \;=\; & -ig_{\beta\mu}\epsilon_{\alpha\nu\rho\sigma} + ig_{\alpha\mu}\epsilon_{\beta\nu\rho\sigma} - ig_{\beta\nu}\epsilon_{\mu\alpha\rho\sigma} + ig_{\alpha\nu}\epsilon_{\mu\beta\rho\sigma} \\ & -ig_{\beta\rho}\epsilon_{\mu\nu\alpha\sigma} + ig_{\alpha\rho}\epsilon_{\mu\nu\beta\sigma} - ig_{\beta\sigma}\epsilon_{\mu\nu\rho\alpha} + ig_{\alpha\sigma}\epsilon_{\mu\nu\rho\beta}\end{aligned}$$

Is it clear this vanishes? It is antisymmetric in μ, ν, ρ, σ, so contracting with $\epsilon^{\mu\nu\rho\sigma}$ will tell us whether it is zero or not. We have

$$\begin{aligned}[L_{\alpha\beta}, \epsilon_{\mu\nu\rho\sigma}]\,\epsilon^{\mu\nu\rho\sigma} &= -ig_{\beta\mu}\epsilon_{\alpha\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - ig_{\beta\nu}\epsilon_{\mu\alpha\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - ig_{\beta\rho}\epsilon_{\mu\nu\alpha\sigma}\epsilon^{\mu\nu\rho\sigma} \\ &\quad -ig_{\beta\sigma}\epsilon_{\mu\nu\rho\alpha}\epsilon^{\mu\nu\rho\sigma} \qquad - (\alpha \leftrightarrow \beta) \\ &= -6ig_{\beta\mu}\delta^\mu_\alpha - 6ig_{\beta\nu}\delta^\nu_\alpha - 6ig_{\beta\rho}\delta^\rho_\alpha - 6ig_{\beta\sigma}\delta^\sigma_\alpha \\ &\quad -(\alpha \leftrightarrow \beta) \\ &= -24ig_{\beta\alpha} - (\alpha \leftrightarrow \beta) = 0.\end{aligned}$$

Thus not changing under commutation with $L_{\alpha\beta}$, which as a c-number is of course what ϵ does, is just what it should do to transform as a totally antisymmetric tensor.

Thus all the components of $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma}$ transform properly and W^μ must as well. Furthermore, $W^2 = W^\mu W_\mu$ must transform like a scalar,

$$[W^\mu W_\mu, L_{\alpha\beta}] = 0.$$

2) Consider a single real free scalar field with the Klein-Gordon Lagrangian.

(a) Find the expression for $T^{\mu\nu}$ in terms of ϕ, π, and $\nabla\phi$.

(b) $\vec{J}(t)$

(c) $\vec{K}(t)$

as integrals over products of $\phi(\vec{x})$ and $\pi(\vec{x})$. [Recall $\vec{J} = (L_{23}, L_{31}, L_{12})$ and $\vec{K} = (L_{01}, L_{02}, L_{03})$.] From these results, find the values of

(d) $[P^\mu(t), \phi(\vec{x}, t)]$,

(e) $[\vec{J}(t), \phi(\vec{x}, t)]$

(f) $[\vec{K}(t), \phi(\vec{x}, t)]$

at equal times, in terms of $\phi(\vec{x})$ and its derivatives.

Solution 2 As we found on the last homework, the transformation of a scalar under translation gives the energy momentum tensor

$$T^{\mu\nu} = (\partial^\mu \phi)\partial^\nu \phi - \mathcal{L} g^{\mu\nu} = (\partial^\mu \phi)\partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \left((\partial^\rho \phi)\partial_\rho \phi - m^2 \phi^2 \right).$$

The conserved charge for translations is the total 4-momentum

$$P^\nu(t) = \int d^3x \, T^{0\nu}(\vec{x}, t),$$

whose zeroth component is, of course the energy or Hamiltonian

$$H = P^0 = \int d^3x \, \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right),$$

and the total 3-momentum is

$$\vec{P}(t) = P^j(t) = -\int d^3x \pi(\vec{x}, t) \nabla \phi(\vec{x}, t).$$

Now we need

$$L^{\nu\rho}(t) = \int d^3x \mathcal{M}^{0\nu\rho}(x),$$

where from the last homework we know

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\rho T^{\mu\nu} - x^\nu T^{\mu\rho},$$

so

$$\begin{aligned} L^{\nu\rho}(t) &= \int d^3x \Big\{ \dot\phi(x) \left(x^\rho \partial^\nu \phi(x) - x^\nu \partial^\rho \phi(x) \right) \\ &\quad - \frac{1}{2} \left(x^\rho g^{0\nu} - x^\nu g^{0\rho} \right) \left(\partial^\sigma \phi(x) \partial_\sigma \phi(x) - m^2 \phi^2(x) \right) \Big\}. \end{aligned}$$

For the rotations, we have

$$L^{ij}(t) = -\int d^3x \, \pi(x) \left(x^j \partial_i \phi(x) - x^i \partial_j \phi(x) \right),$$

where the minus sign comes from lowering the index on the partial derivative to make it an ordinary gradient. For the boosts,

$$\begin{aligned} L^{0j}(t) &= \int d^3x \Big\{ \dot\phi(x) \left(x^j \partial^0 \phi(x) - x^0 \partial^j \phi(x) \right) \\ &\quad - \frac{1}{2} \left(x^j g^{00} - x^0 g^{0j} \right) \left(\partial^\sigma \phi(x) \partial_\sigma \phi(x) - m^2 \phi^2(x) \right) \Big\} \\ &= \int d^3x \Big\{ \pi(x) \left(x^j \pi(x) + t \partial_j \phi(x) \right) \\ &\quad - \frac{1}{2} x^j \left(\pi^2(x) - (\vec\nabla \phi(x))^2 - m^2 \phi^2(x) \right) \Big\} \\ &= \int d^3x \left(\frac{1}{2} x^j \pi^2(x) + t \pi(x) \partial_j \phi(x) + \frac{1}{2} x^j (\vec\nabla \phi(x))^2 + \frac{1}{2} m^2 x^j \phi^2(x) \right). \end{aligned}$$

From the commutation relations (at equal time)

$$[\phi(\vec{x}, t), \phi(\vec{x}\,', t)] = 0, \quad [\phi(\vec{x}, t), \pi(\vec{x}\,', t)] = i\delta^3(\vec{x} - \vec{x}\,'), \quad [\pi(\vec{x}, t), \pi(\vec{x}\,', t)] = 0,$$

we have

$$\begin{aligned} [H, \phi(\vec{y})] &= \int d^3x \left[\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2, \phi(\vec{y}) \right] \\ &= \int d^3x \, \pi(\vec{x}) \left(-i\delta^3(\vec{x} - \vec{y}) \right) = -i\pi(\vec{y}), \\ \left[\vec{P}, \phi(\vec{y}) \right] &= \int d^3x \, \left[-\pi(\vec{x}, t) \nabla\phi(\vec{x}, t), \phi(\vec{y}) \right] \\ &= \int d^3x \left(i\delta^3(\vec{x} - \vec{y}) \right) \nabla\phi(\vec{x}, t) = i\nabla\phi(\vec{y}, t) \\ [J_\ell, \phi(\vec{y})] &= \frac{1}{2}\epsilon_{\ell ij} \left[L^{ij}, \phi(\vec{y}) \right] \\ &= -\frac{1}{2}\epsilon_{\ell ij} \int d^3x \, \left[\pi(x) \left(x^j \partial_i \phi(x) - x^i \partial_j \phi(x) \right), \phi(\vec{y}) \right] \\ &= -\epsilon_{\ell ij} \int d^3x \, [\pi(x), \phi(\vec{y})] \, x^j \partial_i \phi(x) \\ &= +i\epsilon_{\ell ij} \delta^3(\vec{x} - \vec{y}) x^j \partial_i \phi(x) = \epsilon_{\ell ij} y^j \partial_i \phi(y) = i \left(\vec{y} \times \vec\nabla \phi \right)_\ell \\ [K_j, \phi(\vec{y})] &= -\left[L^{0j}, \phi(\vec{y}) \right] \\ &= -\int d^3x \, \Big[\frac{1}{2} x^j \pi^2(x) + t\pi(x) \partial_j \phi(x) + \frac{1}{2} x^j (\vec\nabla \phi(x))^2 \\ &\qquad + \frac{1}{2} m^2 x^j \phi^2(x), \phi(\vec{y}) \Big] \end{aligned}$$

$$
\begin{aligned}
&= -\int d^3x \left[\frac{1}{2}x^j\pi^2(\vec{x}) + t\pi(\vec{x})\partial_j\phi(\vec{x}), \phi(\vec{y})\right] \\
&= i\int d^3x\, \delta^3(\vec{x}-\vec{y})x^j\pi(\vec{x}) + t\partial_j\phi(\vec{x}) \\
&= iy^j\pi(\vec{y}) + iy^0\partial_j\phi(\vec{y})
\end{aligned}
$$