

Homework Solution #3

1) In lecture we defined the Pauli-Lubanski vector

$$W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu L_{\rho\sigma},$$

built of the generators of the Poincaré group. Show that

(a) $[W^\mu, P_\nu] = 0,$

and that its square is a Casimir operator of the Poincaré group, that is

(b) $[W^\mu W_\mu, P_\nu] = 0,$

(c) $[W^\mu W_\mu, L_{\alpha\beta}] = 0.$

Also show, if you haven't already, that P^2 is also a Casimir operator of the group.

It may be useful to examine whether your evaluation of $[L_{\alpha\beta}, W^\mu]$ is what you would expect for a general vector, $[L_{\alpha\beta}, V^\mu]$, and also to show that the vector properties implied by the indices do transform correctly under commutator with $L_{\alpha\beta}$. That is, $V^\mu F_\mu$ should commute, and the c-numbers $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$, which of course do commute with all operators, **should** commute with Lorentz transformations despite having Lorentz indices.

Solution 1 We will need the commutators of L 's and P 's with each other,

$$[L_{\alpha\beta}, L_{\gamma\zeta}] = ig_{\alpha\gamma} L_{\beta\zeta} - ig_{\beta\gamma} L_{\alpha\zeta} - ig_{\alpha\zeta} L_{\beta\gamma} + ig_{\zeta\beta} L_{\alpha\gamma} \quad (1)$$

$$[L_{\alpha\beta}, P_\nu] = -ig_{\beta\nu} P_\alpha + ig_{\alpha\nu} P_\beta \quad (2)$$

$$[P^\mu, P^\nu] = 0 \quad (3)$$

As $W^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P_\alpha L_{\beta\gamma},$

$$[W^\mu, P_\nu] = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P_\alpha [L_{\beta\gamma}, P_\nu] = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} P_\alpha (-ig_{\gamma\nu} P_\beta + ig_{\beta\nu} P_\gamma) = 0 \quad (4)$$

so, of course, $[W^\mu W_\mu, P_\nu] = 0$.

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} [P_\alpha, L_{\nu\rho}] L_{\beta\gamma} + \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha [L_{\beta\gamma}, L_{\nu\rho}] \\
&= \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} (ig_{\rho\alpha}P_\nu - ig_{\nu\alpha}P_\rho) L_{\beta\gamma} \\
&\quad + \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha (ig_{\beta\nu}L_{\gamma\rho} - ig_{\gamma\nu}L_{\beta\rho} - ig_{\beta\rho}L_{\gamma\nu} + ig_{\rho\gamma}L_{\beta\nu})
\end{aligned} \tag{5}$$

Do we believe this? A vector should transform as P does, so we would expect

$$[W^\mu, L_{\nu\rho}] = i\delta_\rho^\mu W_\nu - i\delta_\nu^\mu W_\rho. \tag{6}$$

There are two ways we can verify that, in fact, (6) is correct. The first is to notice that because of the contraction with the epsilon, the α, β, γ indices in the parentheses in (5) may be permuted, changing the sign if necessary, so we can replace $g_{\beta\nu}P_\alpha L_{\gamma\rho}$ and $-g_{\gamma\nu}P_\alpha L_{\beta\rho}$ with $g_{\alpha\nu}P_\gamma L_{\beta\rho}$ and $-g_{\alpha\nu}P_\beta L_{\gamma\rho}$ and also replace $-g_{\beta\rho}P_\alpha L_{\gamma\nu}$ and $g_{\rho\gamma}P_\alpha L_{\beta\nu}$ with $g_{\alpha\rho}P_\beta L_{\gamma\nu}$ and $-g_{\alpha\rho}P_\gamma L_{\beta\nu}$. Thus

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{i}{2}\epsilon^{\mu\alpha\beta\gamma} \left(g_{\alpha\rho} \{ P_\nu L_{\beta\gamma} + P_\beta L_{\gamma\nu} + P_\gamma L_{\nu\beta} \} \right. \\
&\quad \left. - g_{\alpha\nu} \{ P_\rho L_{\beta\gamma} + P_\beta L_{\gamma\rho} + P_\gamma L_{\rho\beta} \} \right)
\end{aligned} \tag{7}$$

The expression in each of the $\{\}$'s is totally antisymmetric in the three indices, and can therefore be written in terms of W , for

$$\epsilon_{\rho\beta\gamma\mu} W^\mu = \frac{1}{2}\epsilon_{\rho\beta\gamma\mu}\epsilon^{\mu\alpha\sigma\tau} P_\alpha L_{\sigma\tau} = P_\rho L_{\gamma\beta} + P_\beta L_{\rho\gamma} + P_\gamma L_{\beta\rho}.$$

Thus we see that

$$\begin{aligned}
[W^\mu, L_{\nu\rho}] &= \frac{i}{2}\epsilon^{\mu\alpha\beta\gamma} (g_{\alpha\rho}\epsilon_{\beta\nu\gamma\tau}W^\tau - g_{\alpha\nu}\epsilon_{\beta\rho\gamma\tau}W^\tau) \\
&= \frac{i}{2}(g_{\alpha\rho}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\tau\nu\beta\gamma}W^\tau + g_{\alpha\nu}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\rho\tau\beta\gamma}W^\tau) \\
&= i(g_{\alpha\rho}(\delta_\tau^\mu\delta_\nu^\alpha - \delta_\tau^\alpha\delta_\nu^\mu)W^\tau + g_{\alpha\nu}(\delta_\rho^\mu\delta_\tau^\alpha - \delta_\rho^\alpha\delta_\tau^\mu)W^\tau) \\
&= ig_{\nu\rho}W^\mu - i\delta_\nu^\mu W_\rho + i\delta_\rho^\mu W_\nu - ig_{\rho\nu}W^\mu \\
&= i\delta_\rho^\mu W_\nu - i\delta_\nu^\mu W_\rho,
\end{aligned} \tag{8}$$

just what it should be!

The second way to find that W^μ transforms by (6) the way it should is to prove our suspicions, that quantities made with vector indices transform the way we think under commutator with $L_{\alpha\beta}$, that is, any vector V_ν obeys

$$[L_{\alpha\beta}, V_\nu] = -ig_{\beta\nu}V_\alpha + ig_{\alpha\nu}V_\beta$$

just as P does in (2). A tensor with two indices should have each one transform that way,

$$[L_{\alpha\beta}, T_{\gamma\zeta}] = -ig_{\beta\gamma}T_{\alpha\zeta} + ig_{\alpha\gamma}T_{\beta\zeta} - ig_{\beta\zeta}T_{\gamma\alpha} + ig_{\alpha\zeta}T_{\gamma\beta}$$

If the tensor is antisymmetric, like L itself,

$$[L_{\alpha\beta}, T_{\gamma\zeta}] = ig_{\alpha\gamma}T_{\beta\zeta} - ig_{\beta\gamma}T_{\alpha\zeta} - ig_{\alpha\zeta}T_{\beta\gamma} + ig_{\beta\zeta}T_{\alpha\gamma},$$

which is Eq. (1) if T is L . So the Lorentz tensors do transform correctly.

Does $g_{\mu\nu}$ transform correctly? As it is a c-number, it commutes with $L_{\alpha\beta}$, but if it transforms as a tensor,

$$[L_{\alpha\beta}, g_{\mu\nu}] = -ig_{\beta\mu}g_{\alpha\nu} + ig_{\alpha\mu}g_{\beta\nu} - ig_{\beta\nu}g_{\mu\alpha} + ig_{\alpha\nu}g_{\mu\beta} = 0$$

as g is symmetric. So we see that not changing is just what it is supposed to do.

Do contractions behave correctly? If V_ν and W_ν transform as P_ν , then

$$\begin{aligned} [L_{\alpha\beta}, V_\nu W^\nu] &= -ig_{\beta\nu}V_\alpha W^\nu + ig_{\alpha\nu}V_\beta W^\nu - i\delta_\beta^\nu V_\nu W_\alpha + i\delta_\alpha^\nu V_\nu W_\beta \\ &= -iV_\alpha W_\beta + iV_\beta W_\alpha - iV_\beta W_\alpha + iV_\alpha W_\beta = 0 \end{aligned}$$

so contracted indices don't contribute to the commutator. This means any compound object we make of components which transform properly by commutation with $L_{\alpha\beta}$ will also do so.

In particular, that means the Minkowski square of any vector which transforms properly under Lorentz transformations will commute with the Lorentz generators, and will be a Casimir operator of the Lorentz group, Thus P^2 , which also commutes with the Poincaré generators P_ν , is a Casimir operator of the Poincaré group.

Now for constructing W^μ we also need the ϵ tensor. For $\epsilon_{\mu\nu\rho\sigma}$

$$\begin{aligned} [L_{\alpha\beta}, \epsilon_{\mu\nu\rho\sigma}] &= -ig_{\beta\mu}\epsilon_{\alpha\nu\rho\sigma} + ig_{\alpha\mu}\epsilon_{\beta\nu\rho\sigma} - ig_{\beta\nu}\epsilon_{\mu\alpha\rho\sigma} + ig_{\alpha\nu}\epsilon_{\mu\beta\rho\sigma} \\ &\quad - ig_{\beta\rho}\epsilon_{\mu\nu\alpha\sigma} + ig_{\alpha\rho}\epsilon_{\mu\nu\beta\sigma} - ig_{\beta\sigma}\epsilon_{\mu\nu\rho\alpha} + ig_{\alpha\sigma}\epsilon_{\mu\nu\rho\beta} \end{aligned}$$

Is it clear this vanishes? It is antisymmetric in μ, ν, ρ, σ , so contracting with $\epsilon^{\mu\nu\rho\sigma}$ will tell us whether it is zero or not. We have

$$\begin{aligned} [L_{\alpha\beta}, \epsilon_{\mu\nu\rho\sigma}] \epsilon^{\mu\nu\rho\sigma} &= -ig_{\beta\mu}\epsilon_{\alpha\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - ig_{\beta\nu}\epsilon_{\mu\alpha\rho\sigma}\epsilon^{\mu\nu\rho\sigma} - ig_{\beta\rho}\epsilon_{\mu\nu\alpha\sigma}\epsilon^{\mu\nu\rho\sigma} \\ &\quad - ig_{\beta\sigma}\epsilon_{\mu\nu\rho\alpha}\epsilon^{\mu\nu\rho\sigma} - (\alpha \leftrightarrow \beta) \\ &= -6ig_{\beta\mu}\delta_\alpha^\mu - 6ig_{\beta\nu}\delta_\alpha^\nu - 6ig_{\beta\rho}\delta_\alpha^\rho - 6ig_{\beta\sigma}\delta_\alpha^\sigma \\ &\quad - (\alpha \leftrightarrow \beta) \\ &= -24ig_{\beta\alpha} - (\alpha \leftrightarrow \beta) = 0. \end{aligned}$$

Thus not changing under commutation with $L_{\alpha\beta}$, which as a c-number is of course what ϵ does, is just what it should do to transform as a totally antisymmetric tensor.

Thus all the components of $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu L_{\rho\sigma}$ transform properly and W^μ must as well. Furthermore, $W^2 = \vec{W}^\mu W_\mu$ must transform like a scalar,

$$[W^\mu W_\mu, L_{\alpha\beta}] = 0.$$

2) Consider a single real free scalar field with the Klein-Gordon Lagrangian.

- (a) Find the expression for $T^{\mu\nu}$ in terms of ϕ , π , and $\nabla\phi$.
- (b) $\vec{J}(t)$
- (c) $\vec{K}(t)$

as integrals over products of $\phi(\vec{x})$ and $\pi(\vec{x})$. [Recall $\vec{J} = (L_{23}, L_{31}, L_{12})$ and $\vec{K} = (L_{01}, L_{02}, L_{03})$.] From these results, find the values of

- (d) $[P^\mu(t), \phi(\vec{x}, t)]$,
- (e) $[\vec{J}(t), \phi(\vec{x}, t)]$
- (f) $[\vec{K}(t), \phi(\vec{x}, t)]$

at equal times, in terms of $\phi(\vec{x})$ and its derivatives.

Solution 2 As we found on the last homework, the transformation of a scalar under translation gives the energy momentum tensor

$$T^{\mu\nu} = (\partial^\mu \phi) \partial^\nu \phi - \mathcal{L} g^{\mu\nu} = (\partial^\mu \phi) \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \left((\partial^\rho \phi) \partial_\rho \phi - m^2 \phi^2 \right).$$

The conserved charge for translations is the total 4-momentum

$$P^\nu(t) = \int d^3x T^{0\nu}(\vec{x}, t),$$

whose zeroth component is, of course the energy or Hamiltonian

$$H = P^0 = \int d^3x \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right),$$

and the total 3-momentum is

$$\vec{P}(t) = P^j(t) = - \int d^3x \pi(\vec{x}, t) \nabla \phi(\vec{x}, t).$$

Now we need

$$L^{\nu\rho}(t) = \int d^3x \mathcal{M}^{0\nu\rho}(x),$$

where from the last homework we know

$$\mathcal{M}^{\mu\nu\rho}(x) = x^\rho T^{\mu\nu} - x^\nu T^{\mu\rho},$$

so

$$\begin{aligned} L^{\nu\rho}(t) = & \int d^3x \left\{ \dot{\phi}(x) (x^\rho \partial^\nu \phi(x) - x^\nu \partial^\rho \phi(x)) \right. \\ & \left. - \frac{1}{2} (x^\rho g^{0\nu} - x^\nu g^{0\rho}) (\partial^\sigma \phi(x) \partial_\sigma \phi(x) - m^2 \phi^2(x)) \right\}. \end{aligned}$$

For the rotations, we have

$$L^{ij}(t) = - \int d^3x \pi(x) (x^j \partial_i \phi(x) - x^i \partial_j \phi(x)),$$

where the minus sign comes from lowering the index on the partial derivative to make it an ordinary gradient. For the boosts,

$$\begin{aligned}
L^{0j}(t) &= \int d^3x \left\{ \dot{\phi}(x) \left(x^j \partial^0 \phi(x) - x^0 \partial^j \phi(x) \right) \right. \\
&\quad \left. - \frac{1}{2} \left(x^j g^{00} - x^0 g^{0j} \right) \left(\partial^\sigma \phi(x) \partial_\sigma \phi(x) - m^2 \phi^2(x) \right) \right\} \\
&= \int d^3x \left\{ \pi(x) \left(x^j \pi(x) + t \partial_j \phi(x) \right) \right. \\
&\quad \left. - \frac{1}{2} x^j \left(\pi^2(x) - (\vec{\nabla} \phi(x))^2 - m^2 \phi^2(x) \right) \right\} \\
&= \int d^3x \left(\frac{1}{2} x^j \pi^2(x) + t \pi(x) \partial_j \phi(x) + \frac{1}{2} x^j (\vec{\nabla} \phi(x))^2 + \frac{1}{2} m^2 x^j \phi^2(x) \right).
\end{aligned}$$

From the commutation relations (at equal time)

$$[\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0, \quad [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}'), \quad [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0,$$

we have

$$\begin{aligned}
[H, \phi(\vec{y})] &= \int d^3x \left[\frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2, \phi(\vec{y}) \right] \\
&= \int d^3x \pi(\vec{x}) \left(-i\delta^3(\vec{x} - \vec{y}) \right) = -i\pi(\vec{y}), \\
[\vec{P}, \phi(\vec{y})] &= \int d^3x \left[-\pi(\vec{x}, t) \nabla \phi(\vec{x}, t), \phi(\vec{y}) \right] \\
&= \int d^3x \left(i\delta^3(\vec{x} - \vec{y}) \right) \nabla \phi(\vec{x}, t) = i\nabla \phi(\vec{y}, t) \\
[J_\ell, \phi(\vec{y})] &= \frac{1}{2} \epsilon_{\ell ij} \left[L^{ij}, \phi(\vec{y}) \right] \\
&= -\frac{1}{2} \epsilon_{\ell ij} \int d^3x \left[\pi(x) \left(x^j \partial_i \phi(x) - x^i \partial_j \phi(x) \right), \phi(\vec{y}) \right] \\
&= -\epsilon_{\ell ij} \int d^3x \left[\pi(x), \phi(\vec{y}) \right] x^j \partial_i \phi(x) \\
&= +i\epsilon_{\ell ij} \delta^3(\vec{x} - \vec{y}) x^j \partial_i \phi(x) = \epsilon_{\ell ij} y^j \partial_i \phi(y) = i \left(\vec{y} \times \vec{\nabla} \phi \right)_\ell \\
[K_j, \phi(\vec{y})] &= - \left[L^{0j}, \phi(\vec{y}) \right] \\
&= - \int d^3x \left[\frac{1}{2} x^j \pi^2(x) + t \pi(x) \partial_j \phi(x) + \frac{1}{2} x^j (\vec{\nabla} \phi(x))^2 \right. \\
&\quad \left. + \frac{1}{2} m^2 x^j \phi^2(x), \phi(\vec{y}) \right]
\end{aligned}$$

$$\begin{aligned} &= - \int d^3x \left[\frac{1}{2} x^j \pi^2(\vec{x}) + t \pi(\vec{x}) \partial_j \phi(\vec{x}), \phi(\vec{y}) \right] \\ &= i \int d^3x \delta^3(\vec{x} - \vec{y}) x^j \pi(\vec{x}) + t \partial_j \phi(\vec{x}) \\ &= i y^j \pi(\vec{y}) + i y^0 \partial_j \phi(\vec{y}) \end{aligned}$$