Physics 615 Nov. 29, 2007
Homework #11 Solutions

1 [15 pts] Do problem 7.3 from Peskin and Schroeder. Note that part of the calculation has already been done in solving Problem 6.3.

Solution 1 (a) We need to consider diagrams with a scalar particle as part of a loop, both as part of the vertex correction to calculate the contribution to $\Gamma^\mu(0) = Z_1^{-1}\gamma^\mu$, and to the electron self energy $\Sigma(p)$ to calculate the contribution to

$$\delta Z_2 = \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p=m}.$$

For the vertex function we found, without using anything specifying the dimension d of space-time, that a scalar particle makes a contribution

$$\delta\Gamma_s^\mu = \frac{i\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\int dx\,dy\,dz\,\delta(x+y+z-1)2\frac{4m^2\gamma^\mu - 4mk^\mu + 2k^\mu\not k - k^2\gamma^\mu}{(\ell^2-\Delta+i\epsilon)^3},$$

where $\ell = k - xp - yp'$ and $\Delta = -xyq^2 + (1-z)^2m^2 + zM_\phi^2$. Expanding the k^μ in terms of ℓ^μ and replacing $\ell^\mu\ell^\nu$ with $g^{\mu\nu}\ell^2/d$, and evaluating at $q=0, p=p'$ (which, by the Gordon identity, says $p^\mu \sim m\gamma^\mu$), we have

$$\begin{aligned}\delta\Gamma_s^\mu(0) &= i\lambda^2\int dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^d\ell}{(2\pi)^d}\\ &\quad\left(\frac{\left(\frac{2-d}{d}\right)\ell^2 + m^2(1+z)^2}{(\ell^2-\Delta+i\epsilon)^3}\right)\gamma^\mu.\end{aligned}$$

Noting that at $q^2=0$, Δ depends only on z and not on x and y separately, and that is also true of the numerator, we can replace

$$\begin{aligned}\int dx\,dy\,dz\,\delta(x+y+z-1)f(z) &\to \int_0^1 dz\,f(z)\int_0^{1-z}dx\\ &= \int_0^1(1-z)f(z)dz.\end{aligned}$$

For the ℓ integrals we have, from the revised "Schwinger trick" notes,

$$\begin{aligned}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta+i\epsilon)^m} &= \frac{(-1)^m i}{(4\pi)^{d/2}}\frac{\Gamma(m-\frac12 d)}{\Gamma(m)}\Delta^{\frac12 d-m} \qquad (1)\\ &\underset{m=3}{\longrightarrow} \frac{-i}{2(4\pi)^{d/2}}\Gamma(3-\frac12 d)\Delta^{\frac12 d-3}\\ \int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta+i\epsilon)^m} &= -\frac{(-1)^m id}{2(4\pi)^{d/2}}\frac{\Gamma(m-1-\frac12 d)}{\Gamma(m)}\Delta^{\frac12 d+1-m} \qquad (2)\\ &\underset{m=3}{\longrightarrow} \frac{id}{4(4\pi)^{d/2}}\Gamma(2-\frac12 d)\Delta^{\frac12 d-2}.\end{aligned}$$

Thus

$$\begin{aligned}\delta Z_1^{-1} &= \frac{-\lambda^2}{2(4\pi)^{d/2}}\int_0^1 dz\,(1-z)\Big[\Gamma(2-\frac12 d)\Delta^{\frac12 d-2}\left(\frac{2-d}{2}\right)\\ &\quad -m^2(1+z)^2\,\Gamma\left(3-\frac d2\right)\Delta^{\frac12 d-3}\Big].\end{aligned}$$

The limit $\epsilon = 4-d\to 0$ may be taken naïvely in the second term, but in the first we need

$$\begin{aligned}&\frac{1}{(4\pi)^{d/2}}\Gamma(\epsilon/2)\Delta^{-\epsilon/2}\left(\frac{-2+\epsilon}{2}\right)\\ &= \frac{-1}{(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma\right)\left(1-\frac{\epsilon}{2}\ln\frac{\Delta}{4\pi}\right)\left(1-\frac{\epsilon}{2}\right)\\ &= \frac{1}{(4\pi)^2}\left[-\frac{2}{\epsilon}+\gamma+1+\ln\left(\frac{\Delta}{4\pi}\right)\right].\end{aligned}$$

Thus

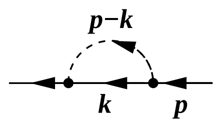
$$\begin{aligned}\delta Z_1^{-1} &= \frac{\lambda^2}{2(4\pi)^2\epsilon} - \frac{\lambda^2}{4(4\pi)^2}(\gamma+1)\\ &\quad -\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dz\,(1-z)\ln\left[\frac{(1-z)^2m^2+zM_\phi^2}{4\pi}\right]\\ &\quad +\frac{\lambda^2m^2}{2(4\pi)^2}\int_0^1 dz\frac{(1-z)(1+z)^2}{(1-z)^2m^2+zM_\phi^2}.\end{aligned}$$

Integrating the first integral by parts, we have

$$\delta Z_1^{-1} = \frac{\lambda^2}{32\pi^2\epsilon} - \frac{\lambda^2}{64\pi^2}(\gamma+1) - \frac{\lambda^2}{64\pi^2}\ln\left(\frac{m^2}{4\pi}\right) - \frac{\lambda^2}{64\pi^2}\int_0^1 dz(1-z)\frac{(1-z)M_\phi^2 - 4(1+z^2)m^2}{(1-z)^2m^2 + zM_\phi^2}.$$

This is the expression we will compare to δZ_2 below.

Now consider the contribution to the electron self-energy from the interaction with the scalar field:



$$\begin{aligned}-i\Sigma_2^{(s)} &= \frac{(-i\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i(\not k+m)}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2-m_\phi^2}\\ &= \frac{\lambda^2}{2}\int\frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\frac{x\not p+m}{(\ell^2-\Delta(x)+i\epsilon)^2}\end{aligned}$$

with $\ell = k - xp$ and $\Delta = -x(1-x)p^2 + xM_\phi^2 + (1-x)m^2$. Using(1),

$$\begin{aligned}-i\Sigma_2^{(s)}(p) &= \frac{i\lambda^2}{2(4\pi)^2}\Gamma(2-\frac{1}{2}d)\int_0^1 (x\not p+m)\,dx\left(\frac{\Delta(x)}{4\pi}\right)^{\frac{1}{2}d-2}\\ &= \frac{i\lambda^2}{2(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma\right)\int_0^1(x\not p+m)\,dx\left[1-\frac{\epsilon}{2}\ln\left(\frac{\Delta(x)}{4\pi}\right)\right]\\ \Sigma_2^{(s)}(p) &= \left(\frac{-\lambda^2}{(4\pi)^2\epsilon}+\frac{\lambda^2\gamma}{2(4\pi)^2}\right)\left(\frac{1}{2}\not p+m\right)\\ &\quad+\frac{\lambda^2}{2(4\pi)^2}\int_0^1(x\not p+m)\,dx\\ &\qquad\ln\left(\frac{-x(1-x)p^2+xM_\phi^2+(1-x)m^2}{4\pi}\right).\end{aligned}$$

Thus

$$\begin{aligned}\delta Z_2 &= \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p=m} = -\frac{\lambda^2}{2(4\pi)^2\epsilon}+\frac{\lambda^2\gamma}{4(4\pi)^2}\\ &\quad+\frac{\lambda^2}{2(4\pi)^2}\int_0^1 x\,dx\,\ln\left(\frac{xM_\phi^2+(1-x)^2m^2}{4\pi}\right)\\ &\quad-\frac{\lambda^2m^2}{(4\pi)^2}\int_0^1 x\,dx\left(\frac{(1-x^2)}{xM_\phi^2+(1-x)^2m^2}\right).\end{aligned}$$

Integrating the first integral by parts, we have

$$\delta Z_2 = -\frac{\lambda^2}{32\pi^2\epsilon}+\frac{\lambda^2\gamma}{64\pi^2}+\frac{\lambda^2}{64\pi^2}\ln\left(\frac{M_\phi^2}{4\pi}\right) - \frac{\lambda^2}{64\pi^2}\int_0^1 dx\frac{2x(2-x-x^2)m^2+x^2M_\phi^2}{xM_\phi^2+(1-x)^2m^2}.$$

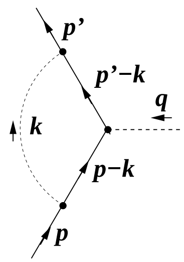
We are asked to show, and the Ward identity demands, that $Z_1 = Z_2$, and as what we have calculated are first order (in λ^2) corrections to zeroth order terms which are both one, this means we should expect $\delta Z_1^{-1} + \delta Z_2 = 0$. Is it?

$$\begin{aligned}\delta Z_1^{-1}+\delta Z_2 &= -\frac{\lambda^2}{64\pi^2}+\frac{\lambda^2}{64\pi^2}\ln\left(\frac{M_\phi^2}{m^2}\right)\\ &\quad-\frac{\lambda^2}{64\pi^2}\int_0^1 dz(1-z)\frac{(1-z)M_\phi^2-4(1+z^2)m^2}{(1-z)^2m^2+zM_\phi^2}\\ &\quad-\frac{\lambda^2}{64\pi^2}\int_0^1 dz\frac{2z(2-z-z^2)m^2+z^2M_\phi^2}{zM_\phi^2+(1-z)^2m^2}\end{aligned}$$

so

$$\begin{aligned}&\frac{64\pi^2}{\lambda^2}\left(\delta Z_1^{-1}+\delta Z_2\right) = -1+\ln\left(\frac{M_\phi^2}{m^2}\right)\\ &\quad-\int_0^1 dz\left(\frac{[(1-z)^2+z^2]M_\phi^2+[-4(1-z)(1+z^2)+2z(1-z)(2+z)]m^2}{(1-z)^2m^2+zM_\phi^2}\right)\\ &= -1+\ln\left(\frac{M_\phi^2}{m^2}\right)\\ &\quad-\int_0^1 dz\left(\frac{(1-2z+2z^2)M_\phi^2-2(1-z)(2-2z+z^2)m^2}{(1-z)^2m^2+zM_\phi^2}\right)\\ &= -1+\ln\left(\frac{M_\phi^2}{m^2}\right)\\ &\quad-\int_0^1 dz\left(\frac{M_\phi^2-2(1-z)m^2}{(1-z)^2m^2+zM_\phi^2}\right)+2\int_0^1 dz(1-z)=0.\end{aligned}$$

(b) Consider the electron vertex with an off-shell scalar field. To lowest order, this is just $-i\frac{\lambda}{\sqrt{2}}\bar{u}(p')u(p)$. The sum of all 1PI diagrams for this vertex is $-i\frac{\lambda}{\sqrt{2}}\bar{u}(p')\Lambda(q^2)u(p)$, where the one loop contributions to Λ come from a scalar or a photon traveling from the incoming to outgoing electron line. Call his contribution $\Lambda_2(q^2)$. Then



$\bar{u}(p')\Lambda_2(q^2)u(p)$

$$\begin{aligned}
&= \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \bar{u}(p') \int \frac{d^dk}{(2\pi)^d} \frac{i(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2 + i\epsilon} \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon} u(p) \\
&\qquad \times \frac{i}{k^2 - M_\phi^2 + i\epsilon} \\
&\quad + (-ie)^2 \bar{u}(p') \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu i(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2 + i\epsilon} \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2 + i\epsilon} \gamma^\nu u(p) \\
&\qquad \times \frac{-ig_{\mu\nu}}{k^2 - \mu^2 + i\epsilon}
\end{aligned}$$

where the first integral comes from exchange of a scalar as shown, and the second from the exchange of a photon, with infrared cutoff mass μ.

Using $\not{p} \sim m$ when acting on $u(p)$, and the same for $\bar{u}(p')\not{p}'$, we have

$$\begin{aligned}
\Lambda_2(q^2) &= i \int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+q-k)^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \\
&\qquad \left(\frac{\lambda^2}{2} \frac{(2m - \not{k})^2}{k^2 - M_\phi^2 + i\epsilon} - e^2 \frac{\gamma^\mu (2m - \not{k})^2 \gamma_\mu}{k^2 - \mu^2 + i\epsilon} \right).
\end{aligned}$$

Let

$$\begin{aligned}
D(M) &= x((p+q-k)^2 - m^2 + i\epsilon) + y((p-k)^2 - m^2 + i\epsilon) \\
&\qquad + (1-x-y)(k^2 - M^2 + i\epsilon) \\
&= k^2 - 2k \cdot ((x+y)p + xq) - (1-x-y)M^2 + i\epsilon \\
&= \ell^2 - \Delta_M + i\epsilon,
\end{aligned}$$

where $k^\mu = \ell^\mu + (x+y)p^\mu + xq^\mu$ and

$$\begin{aligned}
\Delta_M &= (1-x-y)M^2 + (x+y)^2 m^2 + x^2 q^2 + 2x(x+y) p \cdot q \\
&= (1-x-y)M^2 + (x+y)^2 m^2 - xyq^2.
\end{aligned}$$

where I have used $(p+q)^2 - p^2 = 0 = 2p\cdot q + q^2$ in the last line.

We are told looking at the ultraviolet divergent terms will suffice, and due to the six powers of k in the denominators, only the terms with ℓ^2 in the numerator will contribute to the divergence. Using $\gamma^\mu \not{\ell}^2 \gamma_\mu = \ell^2 d$, we have

$$\begin{aligned}
\Lambda_2(q^2) &\sim i \int dx\, dy\, 2\Theta(1-x-y) \int \frac{d^d\ell}{(2\pi)^d} \left(\frac{\lambda^2}{2\Delta_{M_\phi}^3} - \frac{e^2 d}{\Delta_\mu^3} \right) \ell^2 \\
&= \frac{-d}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \int dx\, dy\, 2\Theta(1-x-y) \\
&\qquad \left(\frac{\lambda^2}{2} \left(\frac{\Delta_{M_\phi}}{4\pi} \right)^{-\epsilon/2} - e^2 d \left(\frac{\Delta_{M_\mu}}{4\pi} \right)^{-\epsilon/2} \right) \\
&\sim -\frac{1}{(4\pi)^2\epsilon} (\lambda^2 - 8e^2) \int dx\, dy\, 2\Theta(1-x-y) = -\frac{1}{(4\pi)^2\epsilon} (\lambda^2 - 8e^2).
\end{aligned}$$

[Note: of course we could have used the fact that we are only looking for the quadratically divergent parts earlier, to argue that you can put all external momenta to zero and make masses equal, so you don't need to combine denominators with Feynman.]

This is to be compared to the singular contributions to δZ_2. The contribution from the scalar we have just calculated, $\delta Z_2^{(s)} \sim -\lambda^2/(32\pi^2\epsilon)$. We need to do the photon contribution using dimensional regularization. From 7.17,

$$\begin{aligned}
\Sigma_2^{(\gamma)} &= -ie^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{x\gamma^\mu \not{p} \gamma_\mu + 4m}{(\ell^2 - \Delta + i\epsilon)^2} \\
&= \frac{e^2}{(4\pi)^2} \Gamma(2 - \frac{d}{2}) \int_0^1 dx (-(2-\epsilon)x\not{p} + 4m) \\
&\qquad \left(\frac{-x(1-x)p^2 + x\mu^2 + (1-x)m^2}{4\pi} \right)^{-\epsilon/2} \\
&\sim \frac{e^2}{4\pi^2\epsilon} \int_0^1 dx (x\not{p} + 2m) = \frac{e^2}{8\pi^2\epsilon} (\not{p} + 4m).
\end{aligned}$$

Thus

$$\delta Z_2 = \left. \frac{dZ^2}{d\not{p}} \right|_{\not{p}=m} = \frac{e^2}{8\pi^2\epsilon} - \frac{\lambda^2}{32\pi^2\epsilon} \neq -\Lambda_2 = \frac{1}{(4\pi)^2\epsilon} (\lambda^2 - 8e^2).$$