

Physics 615

Homework #2

Fall, 2007
Due: Sept. 20

These problems are all examples of symmetries treated by Noether's theorem. In evaluating $\delta\mathcal{L}$ it is often easier to use the expression

$$\delta\mathcal{L} = (\partial_\mu \delta x^\mu) \mathcal{L} + \delta\phi_i \frac{\partial\mathcal{L}}{\partial\phi_i} + (\delta\partial_\mu\phi_i) \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_i} + \delta x^\mu \frac{\delta\mathcal{L}}{\delta x^\mu}$$

rather than the expression involving $\delta\phi_i$ and $\frac{\partial\mathcal{L}}{\partial x^\mu}$.

- 1.) Consider the Lagrangian for a set of free real scalar fields $\phi_i, i = 1 \dots N$,

$$\mathcal{L} = \frac{1}{2} \sum_i \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi^2, \quad (1)$$

where $\phi^2 := \sum_i \phi_i^2$. Consider the internal symmetry transformation

$$\delta x^\mu = 0, \quad \delta\phi_i = \epsilon A_{ij} \phi_j,$$

where A_{ij} is an antisymmetric real matrix and we are assuming the summation convention that repeated indices are summed over. Show that this is a symmetry by calculating $\delta\mathcal{L}$, and find the conserved current J^μ and the charge Q corresponding to a given infinitesimal ϵA_{ij} . This is a global $SO(N)$ symmetry.

- 2.) For the case $N = 2$ above, there is only one symmetry generator in $SO(2)$, with $A_{12} = \epsilon = -A_{21}, A_{11} = A_{22} = 0$. Define ϕ without an index to be the complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, as we did on homework #1. Express the current and the conserved charge in terms of ϕ and ϕ^\dagger .
- 3.) Consider the Lagrangian (1) of problem 1, but now consider symmetry under translations,

$$\delta x^\mu = \epsilon a^\mu (\text{constant}), \quad \delta\phi_i = 0.$$

Show this is a symmetry, and find the conserved current $J^\mu = a_\nu T^{\nu\mu}$, which gives you the energy momentum, or stress-energy, tensor $T^{\nu\mu}$. Write $T^{\nu\mu}$, \vec{P} , and H in terms of the fields ϕ_i .

- 4.) Again consider the Lagrangian (1) of problem 1, but now for a Lorentz transformation $\delta x^\mu = \epsilon L^\mu{}_\nu x^\nu$, with $L_{\mu\nu} = -L_{\nu\mu}$ as befits the generator of a Lorentz transformation. The current is then¹ $J^\mu = \frac{1}{2} L_{\nu\rho} \mathcal{M}^{\mu\nu\rho}$. Find the expression for $\mathcal{M}^{\mu\nu\rho}$ in terms of the stress-energy tensor.
- 5.) Once again consider the Lagrangian (1) of problem 1, but now for a dilation $\delta x^\mu = \epsilon x^\mu$. Can you find a symmetry of this form in general or for some particular m ? If so, find the appropriate $\delta\phi$, the current and the charge.

¹The one-half is inserted because the summation convention includes both L_{12} and L_{21} , though these are equivalent, with only the $L_{\mu\nu}$ with $\mu < \nu$ independent. We therefore also require $\mathcal{M}^{\mu\nu\rho} = -\mathcal{M}^{\mu\rho\nu}$.