

Physics 615

Homework #1

Fall, 2007
Due: Sept. 13

1 Consider the Lagrangian for two free real scalar fields ϕ_j with equal masses, with

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^2 \partial_\mu \phi_j \partial^\mu \phi_j - \frac{1}{2} m^2 \sum_{j=1}^2 \phi_j^2. \quad (1)$$

Define ϕ without an index to be the complex field

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}},$$

and rewrite the Lagrangian in terms of ϕ and ϕ^\dagger , its complex conjugate. [Generally we use \dagger for Hermitean conjugate, which is more appropriate if we consider ϕ as an operator, but here Hermitean and complex conjugate are synonymous.] Find the equations of motion for ϕ_j , and show that one gets the same results if we treat $\mathcal{L}(\phi, \phi^\dagger)$ as if ϕ and ϕ^\dagger were independent real fields. Also find the Hamiltonian density both in terms of ϕ_i and by treating ϕ and ϕ^\dagger as if they were independent, and compare.

2 Maxwell's theory of electromagnetism can be expressed in terms of a 4-vector field A_μ , with a Lagrangian density given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where the field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and with the identification with usual Maxwell fields given by $E^i = F^{i0}$, $\epsilon_{ijk} B^k = -F^{ij}$. Note that the degrees of freedom are A_μ , not $F_{\mu\nu}$.

(a) Derive the equations of motion for A_μ , and reexpress them in terms of the electric and magnetic fields.

- (b) Find the canonical momenta Π_μ conjugate to A^μ , and give their constitutive equations. Can these be inverted to solve for \dot{A}^μ as functions of A^ν and Π^ν ?

[Note: Π^μ are not the generalizations π^μ which we defined with the $\bar{\delta}$ for each field. Here the index μ refers to which field, A^μ , we are finding the conjugate to.]

3 The last Problem described a free electromagnetic field in the absence of charges and currents. To include such sources, add a term $-A_\mu j^\mu$ to the lagrangian density, so

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu,$$

where $j^0 = \rho$ is the charge density, and $j^i = \vec{j}$ is the current density, in Heaviside units.

- (a) Derive the new equations of motion for A_μ , and reexpress them in terms of the electric and magnetic fields.
- (b) Show that the equations of motion are only consistent if the source satisfies

$$\partial_\mu j^\mu = 0.$$