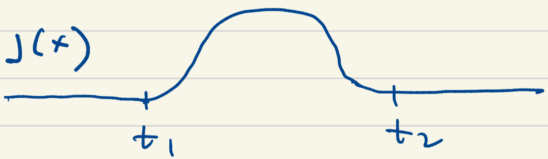
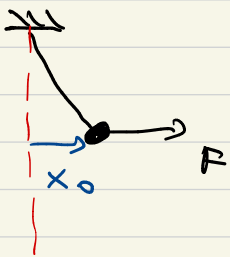
Particle creation by a classical source



$$(\partial^2 + m^2)\,\phi(x) = J(x) \leftarrow \text{source term}$$

Simple harmonic oscillator: $\ddot{\phi} + \omega^2 \phi = F$



$$x \to x + x_0$$

$$x = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \implies a \to a + \alpha, \quad a^\dagger \to a^\dagger + \alpha^*$$

$$\langle (a^\dagger + \alpha^*)(a + \alpha) \rangle = |\alpha|^2, \quad \Delta E = \omega |\alpha|^2$$

$$(\partial^2+m^2)\,\phi(x) = J(x) \implies \mathcal{L} = \frac{(\partial_\mu \phi)^2}{2} - \frac{m^2\phi^2}{2} + J(x)\,\phi(x)$$

Solution of the homogeneous field eq. $[\,J(x)=0\,]$

$$\phi_0(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{ip\cdot x} \right)$$

Retarded Green's fn: $\quad (\partial^2+m^2)\,D_R(x-y)\,J(y) = -i\,\delta^{(4)}(x-y)\,J(y)$

$$\int \delta^{(4)}(x-y)\,J(y)\,d^4y = J(x)$$

$$(\partial^2+m^2)\int D_R(x-y)\,J(y)\,d^4y = -i\,J(x)$$

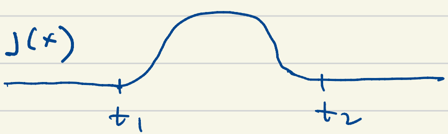
$$\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\,J(y)$$

$$D_R(x-y) = \theta(x^0-y^0)\langle[\varphi(x),\varphi(y)]\rangle = \theta(x^0-y^0)\left[D(x-y) - D(y-x)\right] =$$

$$= \theta(x^0-y^0)\int_{\vec{p}} \frac{e^{-ip\cdot(x-y)} - e^{ip(x-y)}}{2E_{\vec{p}}}$$

$$\phi(x) = \phi(x_0) + i\int d^4y\, D_R(x-y)\, J(y)$$

$$\phi(x) = \phi_0(x) + i\int d^4y \int_{\vec{p}} \frac{J(y)}{2E_{\vec{p}}}\,\theta(x^0-y^0)\times\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$



Let $x^0 > t_2$

$$\tilde{J}(p) = \int d^4y\, J(y)\, e^{ip\cdot y}$$

$$\phi_0(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\, e^{-ip\cdot x} + a^{+}_{\vec{p}}\, e^{ip\cdot x}\right)$$

$$\phi(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} \left\{ \left[a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{J}(p) \right] e^{-ip\cdot x} + c.c. \right\}$$

$$\phi_0(x) = \int_{\vec{p}} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a^{+}_{\vec{p}} e^{ip\cdot x} \right)$$

$$a_p \longrightarrow a_p + \frac{i}{\sqrt{2E_p}} \tilde{J}(p)$$

$$H = \int_{\vec{p}} E_p \left(a^{+}_p - \frac{i}{\sqrt{2E_p}} \tilde{J}(p) \right) \left(a_p + \frac{i}{\sqrt{2E_p}} \tilde{J}^{*}(p) \right)$$

$$\langle 0 | H | 0 \rangle = \int_{\vec{p}} \frac{|\tilde{J}(p)|^2}{2} = \int_{\vec{p}} E_p \frac{|\tilde{J}(p)|^2}{2E_p}$$

$$\# \text{ of particles produced} = \int_{\vec{p}} \frac{|\tilde{J}(p)|^2}{2E_p}$$

KG eq. $(\partial^\mu \partial_\mu + m^2)\phi = 0$ Lorentz inv

$\partial^\mu \partial_\mu$, m^2, ϕ - scalar

Maxwell eqs $\partial^\mu F_{\mu\nu} = 0$ $\partial^\mu \mathcal{F}_{\mu\nu} = \frac{4\pi}{c} J_\nu$

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Lorentz transformations = boosts + 3D rotations

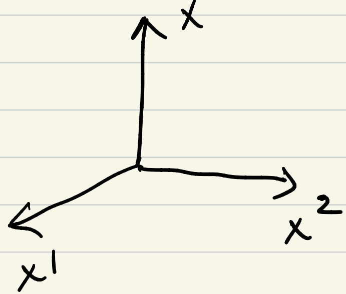
Linear $\quad x' = \Lambda x \qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

$x \xrightarrow[\Lambda]{} x' \xrightarrow[\Lambda']{} x'' \qquad \Lambda'' = \Lambda' \Lambda$ – Lorentz group

$x \xrightarrow[\Lambda'']{} x''$

Rotations

3 rotations



$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_0' = \gamma(x_0 + \beta x_1) \qquad x_1' = \gamma(x_1 + \beta x_0)$$

$$\beta = v \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$x_0^2 - x_1^2$$

$$x_0^2 + (i x_1)^2$$

$$\underbrace{\left(\frac{1}{\sqrt{1-v^2}}\right)^2}_{\cosh\theta \,=} - \underbrace{\left(\frac{v}{\sqrt{1-v^2}}\right)^2}_{\sinh\theta\,=} = 1$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\theta \to i\theta$$

$$\begin{pmatrix} x_0' \\ ix_1' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_0 \\ ix_1 \end{pmatrix}$$

$$\cosh(i\theta) = \cos\theta$$

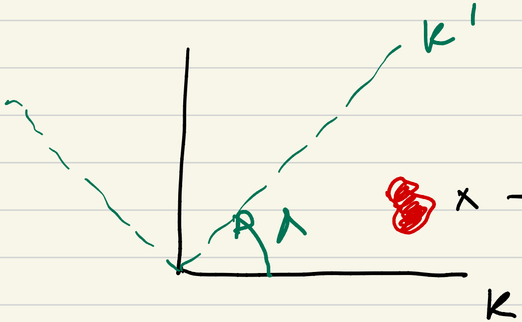
$$\sinh(i\theta) = i\sin\theta$$

Now 6 indep. rotations

$x^0 x^1 \quad x^0 x^2 \quad x^0 x^3 \quad x^1 x^2 \quad x^1 x^3 \quad x^2 x^3$

$x' = \Lambda x \qquad x' = \Lambda^{-1} x$

$\phi'(x') = \phi(x)$



$x \to x' = \Lambda x$

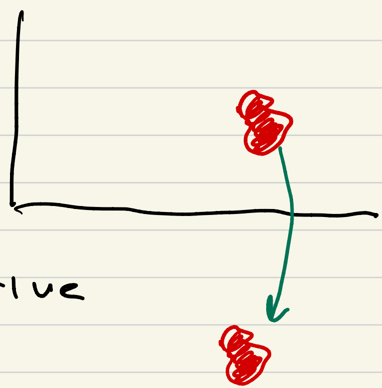
passive

$\phi'(\Lambda x) = \phi(x)$

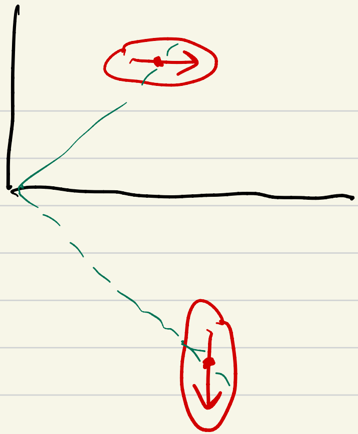
passive

$\phi'(x) = \phi(\Lambda^{-1} x)$

active



active



$$x' = R x$$

$$V^i(x) \longrightarrow R^{ij}\, V^j(R^{-1} x)$$

Lorentz $\quad V^\mu(x) \to \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1} x)$

$$T^{\mu\nu b\rho s\dots} = 0$$

What Φ_a — n-component multiplet?

$$\Phi_a(x) \longrightarrow M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1} x)$$

$$x \xrightarrow[\Lambda]{M(\Lambda)} x' \xrightarrow[\Lambda']{M(\Lambda')} x''$$

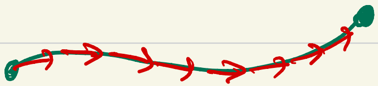
$$x \xrightarrow[\Lambda'']{M(\Lambda'')} x''$$

$$M(\Lambda'') = M(\Lambda')\, M(\Lambda)$$

$$\Lambda'' = \Lambda' \Lambda$$

M — n-dim rep of Lorentz group Λ

Continuous group



Consider translations in 3D

$$f(\vec{x}) \to f(\vec{x} + \vec{a})$$

small $\vec{a}$

$$f(\vec{x} + \vec{a}) = f(\vec{x}) + \underbrace{\vec{a} \cdot \nabla}_{i \, \vec{a} \cdot \hat{\vec{p}}} f =$$

$$\hat{\vec{p}} = -i \nabla$$

$$= \left(1 + i \, \vec{a} \cdot \vec{p}\right) f$$

$$f(\vec{x} + \vec{a}) = \left(1 + \vec{a} \cdot \nabla + \frac{(\vec{a} \cdot \nabla)^2}{2} + \dots\right) f(\vec{x})$$

$$f(\vec{x}+\vec{a}) = \left(1+\vec{a}\cdot\nabla + \frac{(\vec{a}\cdot\nabla)^2}{2} + \dots\right) f(\vec{x}) =$$

$$= e^{\vec{a}\cdot\nabla} f(\vec{x}) = e^{i\vec{a}\cdot\vec{p}} f(\vec{x})$$

$$i\vec{p}\cdot\vec{a}$$

$$\hat{h} \qquad e^{i\alpha\hat{h}}$$