Last time:

\[ \mathcal{L} = \frac{\pi^2}{2} + \frac{\mu^2 \phi^2}{2} + \frac{\lambda}{4!} \phi^4 \quad \Rightarrow \quad \mathcal{L} = \frac{\left(\partial \mu \phi \right)^2}{2} - \frac{\mu^2 \phi^2}{2} - \frac{\lambda}{4!} \phi^4 \]

\( \phi^4 \) - fourth theory. Simplest interacting QFT.
\[ \mathcal{L} = \frac{\pi^2}{2} + \frac{w^2 \phi^2}{2} + \frac{\lambda}{4!} \phi^4 \]

\[ \mathcal{L} = \left( \partial \phi \right)^2 - \frac{w^2 \phi^2}{2} - \frac{\lambda}{4!} \phi^4 \]

Phi-fourth theory. Simplest interacting QFT.

Today: Perturbation expansion of correlation functions

2-point correlation function = 2-point Green's function in \( \phi^4 \) theory

\[ \langle \phi(x) \phi(y) \rangle \rightarrow \text{amplitude of propagation between } x \text{ and } y \]

\[ | \langle 0 \rangle | \rightarrow \text{gr. st. of int. theory} \]

Free theory: \[ \langle 0 | \mathcal{T} \phi(x) \phi(y) | 0 \rangle = \mathcal{D}_\text{free}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i p \cdot (x-y)}}{p^2 - w^2 + i\varepsilon} \]

Now: \[ H = H_0 + H_{\text{int}} = H_K + \int d^3x \, \frac{\lambda}{4!} \phi^4(x) \]
Hint enters in 1. \( \phi(t) = e^{i H t} \phi(0) e^{-i H t} \)

2. \( |\omega\rangle \)

Let's begin with \( \phi(x) \) at fixed time \( t_0 \):

\[
\phi(t_0, \vec{x}) = \int \frac{1}{\sqrt{2\pi}\hbar} \left( a_\phi e^{i \vec{p} \cdot \vec{x}} + a_\phi^\dagger e^{-i \vec{p} \cdot \vec{x}} \right)
\]

\( t \neq t_0 \) \( \phi(t, \vec{x}) = e^{i H (t - t_0)} \phi(t_0, \vec{x}) e^{-i H (t - t_0)} \)

For \( \lambda = 0 \) \( H \rightarrow H_0 \):

\[ \phi(t, \vec{x}) \bigg| _{\lambda = 0} = e^{i H_0 (t - t_0)} \phi(t_0, \vec{x}) e^{-i H_0 (t - t_0)} = \phi_{\lambda = 0} (t, \vec{x}) \]

\[ \downarrow \]

14t. picture field
\[ \phi_\pm(t, x) = \int \frac{1}{\sqrt{2\pi}} \left( \phi_p^- e^{-i p \cdot x} + \phi_p^+ e^{i p \cdot x} \right) \bigg|_{x_0 = t - t_0} \]

\[ \phi(t, x^2) = e^{c H(t - t_0)} e^{-i H_0(t - t_0)} \phi_\pm(t, x) e^{c H_0(t - t_0)} e^{-i H(t - t_0)} = \]

\[ U^+(t, t_0) \phi_\pm(t, x) U(t, t_0) \]

\[ U(t, t_0) = e^{c H_0(t - t_0)} e^{-i H(t - t_0)} \]

"\textit{U}(t, t_0) \text{ or time evol. opt} \]
$$U(t_1, t_0) = e^{c\, H_0 (t_1 - t_0)} e^{-i\, H (t_1 - t_0)}$$

$$i \frac{\partial}{\partial t} U(t_1, t_0) = e^{c\, H_0 (t_1 - t_0)} (-H_0 + H) e^{-i\, H (t_1 - t_0)}$$

$$\Rightarrow U(t_1, t_0) = e^{c\, H_0 (t_1 - t_0)} e^{-i\, H_0 (t_1 - t_0)} = \phi_I(x)$$

$$e^{c\, H_0 (t_1 - t_0)} \Psi(t_0, x) e^{-i\, H_0 (t_1 - t_0)} = \Psi(x)$$

$$i \frac{\partial}{\partial t} U(t_1, t_0) = H_I (t) \cdot U(t_1, t_0)$$
\[ \frac{\partial}{\partial t} u(t, t_0) = H_+ (t) u(t, t_0) \]

\[ \frac{\partial}{\partial t} u = (-i) H_{\pm} u \]

\[ u(t_0, t_0) = 1 \]

0. 0-th order \( u(t, t_0) = 1 \)

1. 1st order \( \frac{\partial}{\partial t} u = (-i) H_{\pm} \)

\[ u(t, t_0) = 1 + (-i) \int_{t_0}^{t} d\tau_1 H_{\pm}(\tau_1) \]

2. 2nd order \( \frac{\partial}{\partial t} u = (-i) H_{\pm} + (-i)^2 H_{\pm}(t) \int_{t_0}^{t} d\tau_1 H_{\pm}(\tau_1) \)

\[ u(t_1, t_0) = 1 + (-i) \int_{t_0}^{t} H_{\pm}(\tau_1) d\tau_1 + (-i)^2 \int_{t_0}^{t} \int_{t_0}^{t_1} d\tau_2 H_{\pm}(\tau_1) H_{\pm}(\tau_2) \]
\[ u(t_{1},t_{0}) = 1 + (-\varepsilon)^{1} \int_{t_{0}}^{t} H_{2}(t_{1}) dt_{1} + (-\varepsilon)^{2} \int_{t_{0}}^{t} \int_{t_{0}}^{t_{1}} H_{2}(t_{1}) H_{1}(t_{2}) dt_{1} dt_{2} + \ldots \]

\[ + \ldots (-\varepsilon)^{n} \int_{t_{0}}^{t} \int_{t_{0}}^{t_{1}} \ldots \int_{t_{0}}^{t_{n-1}} H_{2}(t_{1}) H_{1}(t_{2}) \ldots H_{1}(t_{n}) dt_{1} dt_{2} \ldots dt_{n} \]
\[
\int_{t_0}^{t_1} \int_{t_0}^{t_2} H_+^2(t_1) H_\mp^2(t_2) = \frac{1}{2} \int_{t_0}^{t_1} \int_{t_0}^{t_2} T H_+^2(t_1) H_\mp^2(t_2)
\]

Similarly
\[
\int_{t_0}^{t_1} \int_{t_2}^{t_3} \cdots \int_{t_{n-1}}^{t_n} H_\pm(t_1) \cdots H_\pm(t_n) = \frac{1}{n!} \int_{t_0}^{t_1} \int_{t_2}^{t_3} \cdots \int_{t_{n-1}}^{t_n} \mathcal{T} H_\pm(t_1) \cdots H_\pm(t_n)
\]

\[
U(t_1, t_0) = 1 + (-i) \int_{t_0}^{t_1} H(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^{t_1} \int_{t_2}^{t_3} T H_\pm(t_1) H_\pm(t_2) + \cdots
\]

\[
\ldots = T \exp \left[ -i \int_{t_0}^{t} H_\pm(t) \right]
\]

\[
U(t, t') = T \exp \left[ -i \int_{t'}^{t} H_\pm(t') \right]
\]

\[
i \frac{\partial}{\partial t} U(t, t') = H_\pm(t) U(t, t') \quad \text{with} \quad \partial_+ = t'
\]

\[
U = 1
\]

\[ u(t, t') = e^{i \mathcal{H}_0 (t - t_0)} e^{-i \mathcal{H} (t - t')} e^{-i \mathcal{H}_0 (t' - t_0)} \]

\[ u(t, t_0) = e^{i \mathcal{H}_0 (t - t_0)} e^{-i \mathcal{H} (t - t_0)} \]

\[ i \frac{\partial}{\partial t} u(t, t') = \mathcal{H}_0 (t) \, u(t, t') \]

\[ u(t', t_0) = e^{i \mathcal{H}_0 (t' - t_0)} e^{-i \mathcal{H} (t' - t_0)} \]

\[ u(t, t') = e^{i \mathcal{H}_0 (t - t_0)} e^{-i \mathcal{H} (t - t')} e^{-i \mathcal{H}_0 (t' - t_0)} \]