1 Physics 613: Problem Set 5 (due Friday April 8)

1.1 Spin operator revisited

In our derivation of the electron magnetic moment from QED, we had the formula

\[ \langle e | H_{\text{eff}} | e \rangle = -\frac{eB}{m} \langle e | \bar{\Psi} S^{12} \Psi | e \rangle \]  

(1)

where \( S^{12} \) came from \( S_{\mu\nu} \equiv \frac{i}{4} [\gamma_{\mu}, \gamma_{\nu}] \). I said in class that \( S^{12} \) was the generator of spin in the \( z \) direction, i.e. \( S^{12} = S_z \). Using the explicit form of the \( \gamma \) matrices, verify more generally that \( S^{ij} = \frac{1}{2} \epsilon^{ijk} S_k \).

1.2 Lie algebra facts

1. Prove that \([T^a, T^b] \) is traceless and anti-Hermitian.

2. Prove that the structure constants \( f^{abc} \) are totally antisymmetric.

3. Prove the Jacobi identity \([T^a, [T^b, T^c]] + [T^c, [T^a, T^b]] + [T^b, [T^c, T^a]] = 0 \)

4. Prove that the adjoint commutation relations \([T^a_{\text{adj}}, T^b_{\text{adj}}] = i f^{abc} T^c_{\text{adj}} \) are equivalent to the Jacobi identity.

5. Prove that \([T^a T^a, T^b] = 0 \) (therefore \( T^a T^a \) must be proportional to the identity in every representation; the proportionality constant is called the Casimir invariant).

1.3 SU(3) generators

Look up the Gell-mann basis of the \( SU(3) \) generators and explicitly find all the nonzero structure constants for \( SU(3) \) in this basis (please use a computer, e.g. Mathematica, for this, don’t do it by hand as it would be way too tedious!)

1.4 SU(3) representations

\( SU(3) \) irreducible representations (irreps) are labeled by two integers \((n, m)\) and can be thought of as all multiple-index tensors of the form \( A_{j_1...j_n}^{i_1...i_m} \), with indices running from 1, 2, 3, which are totally symmetric in all upper and all lower indices and are traceless under contraction of any upper with any lower index. For example, the fundamental rep corresponds to \((1, 0)\) and the anti-fundamental to \((0, 1)\). The adjoint corresponds to \((1, 1)\).
1. Show that \( \dim(n, 0) = \dim(0, n) = \frac{1}{2}(n + 2)(n + 1) \).

Irreps can be multiplied (tensor product) and decomposed (direct sum) into smaller irreps by symmetrizing and tracing over indices.

For example we can multiply a fundamental and antifundamental \( A^a B_b = C^a_b + \frac{1}{3} \delta^a_b A^c B_c \) where \( C \) is the traceless part of \( AB \). In this way we obtain the sum of the adjoint (octet) and the trivial (singlet) representation. We can express this more mathematically as \( (1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0) \), or in terms of the dimensions, \( 3 \otimes 3 = 8 \oplus 1 \).

One can derive (it takes a bit of thought) the following multiplication rule \( (n, 0) \otimes (m, 0) = (n + m, 0) \oplus (n + m - 2, 1) \oplus (n + m - 4, 2) \oplus \ldots \) and similarly for \( (0, n) \otimes (0, m) \).

2. The \((u, d, s)\) quarks transform in the fundamental of \( SU(3) \) and their antiquarks transform in the anti-fundamental of \( SU(3) \). Using the facts above, show that the mesons (which are quark anti-quark bound states) must transform in either the adjoint (octet) or the singlet representation.

3. Using the facts above, show that the baryons (which are 3-quark bound states) must transform in either the adjoint (octet) or the decuplet representation.