

Some Spherical Models

Uniform density sphere

$$\rho = \text{const.} \quad m(r) = M \frac{r^3}{a^3} \quad \Phi(r) = \frac{GM}{2a^3}(r^2 - 3a^2) \quad v_c^2(r) = \frac{GM}{a^3}r^2.$$

if $r \leq a$

Plummer sphere (Plummer, H. C. 1911, *MNRAS*, **71**, 460)

$$\rho = \frac{3M}{4\pi a^3}(1+x^2)^{-5/2} \quad m(r) = \frac{Ma^2}{(1+x^2)^{3/2}} \quad \Phi(r) = -\frac{GM}{a}(1+x^2)^{-1/2}$$

$$v_c^2(r) = \frac{GM}{a} \frac{x^2}{(1+x^2)^{3/2}}.$$

$x = r/a$

Spherical isochrone (Henon, M. 1959, *Ann. d'Ap*, **22**, 126 in french)

$$\rho = \frac{Ma(a+2r_*)}{4\pi r_*^3(a+r_*)^2} \quad m(r) = \frac{Mr^3}{r_*(a+r_*)^2} \quad \Phi(r) = -\frac{GM}{a+r_*} \quad v_c^2(r) = \frac{GM}{r_*} \frac{r^2}{(a+r_*)^2}.$$

$r_* = \sqrt{a^2 + r^2}$.

Singular isothermal sphere

$$\rho = \frac{M}{4\pi a r^2} \quad m(r) = M \frac{r}{a} \quad \Phi(r) = GM \ln\left(\frac{r}{a}\right) \quad v_c^2(r) = \frac{GM}{a} = \text{const.}$$

a is a reference radius at which the mass interior is M .

Hernquist model (Hernquist, L. 1990, *Ap. J.*, **356**, 359)

$$\rho = \frac{Ma}{2\pi r(r+a)^3} \quad m(r) = \frac{Mr^2}{(a+r)^2} \quad \Phi(r) = -\frac{GM}{a+r} \quad v_c^2(r) = \frac{GMr}{(r+a)^2}.$$

Jaffe model (Jaffe, W. 1983, *MNRAS*, **202**, 995)

$$\rho = \frac{Ma}{4\pi r^2(a+r)^2} \quad m(r) = \frac{Mr}{a+r} \quad \Phi(r) = GM \ln\left(\frac{r}{r+a}\right) \quad v_c^2(r) = \frac{GM}{r+a}.$$

γ models (Dehnen, W. 1993, *MNRAS*, **265**, 250)

$$\rho = \frac{Ma(3-\gamma)}{4\pi r^\gamma(a+r)^{4-\gamma}} \quad m(r) = M \left(\frac{r}{a+r}\right)^{3-\gamma} \quad \Phi(r) = -\frac{GM}{a(2-\gamma)} \left[1 - \left(\frac{r}{r+a}\right)^{2-\gamma}\right]_{\gamma \neq 2}$$

$$v_c^2(r) = \frac{GMr^{2-\gamma}}{(r+a)^{3-\gamma}}.$$

$0 \leq \gamma < 3$, $\gamma = 2$ is Jaffe's model, $\gamma = 1$ is Hernquist's model. Tremaine *et al.* (1994, *AJ*, **107**, 634) use $\eta = 3 - \gamma$.

NFW model (Navarro, J. F., Frenk, C. S. & White, S. D. M. 1997, *Ap. J.*, **490**, 493)

$$\rho = \frac{\rho_s r_s^3}{r(r+r_s)^2} \quad m(r) = M_* \left[\ln(1+x) - \frac{x}{1+x} \right] \quad \Phi(r) = -\frac{GM}{r_s} \frac{\ln(1+x)}{x}$$

$$v_c^2(r) = \frac{GM}{r_s} \left[\frac{x}{1+x} - \frac{\ln(1+x)}{x} \right],$$

where $M_* = 4\pi\rho_s r_s^3$, $x = r/r_s$.

Some Disk Models

Maclaurin-Freeman-Kalnajs disk

$$\Sigma(R) = \frac{3M}{2\pi a^2} \sqrt{1-x^2} \quad m(R) = M \left[1 - (1-x^2)^{3/2} \right] \quad \Phi(R) = -\frac{3\pi GM}{4a} \left(1 - \frac{x^2}{2} \right)$$

$$v_c^2(R) = \frac{3\pi GM}{4a} x^2$$

where $x = R/a$ and $x \leq 1$. The circular speed in the disk plane, which peaks at the disk edge at a value greater than is possible for a spherical distribution of the same mass, declines faster than Kepler just beyond the edge of the disk.

Kuz'min-Toomre disk (Toomre, A. 1963, *Ap. J.*, **138**, 385)

$$\Sigma(R) = \frac{M}{2\pi a^2} (1+x^2)^{-3/2} \quad m(R) = M \left[1 - (1+x^2)^{-1/2} \right] \quad \Phi(R) = -\frac{GM}{a} (1+x^2)^{-1/2}$$

$$v_c^2(R) = \frac{GM}{a} \frac{x^2}{(1+x^2)^{3/2}}$$

where $x = R/a$.

Exponential disk (Freeman, K. C. 1970, *Ap. J.*, **160**, 811)

$$\Sigma(R) = \frac{M\alpha^2}{2\pi} e^{-\alpha R} \quad m(R) = M \left[1 - (1 + \alpha R)e^{-\alpha R} \right]$$

$$\Phi(R) = -\frac{1}{2} GM\alpha^2 R \left[I_0\left(\frac{\alpha R}{2}\right) K_1\left(\frac{\alpha R}{2}\right) - I_1\left(\frac{\alpha R}{2}\right) K_0\left(\frac{\alpha R}{2}\right) \right]$$

$$v_c^2(R) = \frac{1}{2} GM\alpha^2 R \left[I_0\left(\frac{\alpha R}{2}\right) K_0\left(\frac{\alpha R}{2}\right) - I_1\left(\frac{\alpha R}{2}\right) K_1\left(\frac{\alpha R}{2}\right) \right],$$

where $\alpha = 1/R_d$ and I_n and K_n are modified Bessel functions of the first and second kinds.

Gaussian disk (Toomre, A. 1963, *Ap. J.*, **138**, 385)

$$\Sigma(R) = \frac{M}{2\pi\sigma^2} e^{-R^2/(2\sigma^2)} \quad m(R) = M \left[1 - e^{-R^2/(2\sigma^2)} \right]$$

The circular speed in this disk can be written in terms of confluent hypergeometric functions, but the potential is known only numerically.

Isochrone disk (Kalnajs, A. J. 1976, *Ap. J.*, **205**, 751)

$$\Sigma(R) = \frac{M}{2\pi a^2} \left[\ln \frac{R + R_*}{a} - \frac{R}{R_*} \right] \quad m(R) = \quad \Phi(R) = -\frac{GM}{a + R_*}$$

$$v_c^2(R) = GM \frac{R^2}{R_*(a + R_*)^2}$$

$$R_* = \sqrt{R^2 + a^2}$$

Mestel-Toomre-Zang disk

$$\Sigma(R) = \frac{M}{2\pi a R} \quad m(R) = M \frac{R}{a} \quad \Phi(R) = GM \ln \frac{R}{a} \quad v_c^2(R) = \frac{GM}{a} = \text{const.},$$

where a is a reference radius at which the mass interior is M .

Finite Mestel disk (Mestel, L. 1963, *MNRAS*, **126**, 553)

$$\Sigma(R) = \frac{M}{4aR} \left[1 - 2 \frac{\arcsin(x)}{\pi} \right] \quad m(R) = M \left\{ x \left[\frac{\pi}{2} - \arcsin(x) \right] + 1 - \sqrt{1 - x^2} \right\}$$
$$v_c^2(R) = \frac{\pi GM}{2a}$$

$x = R/a$ & $x < 1$, and the potential requires numerical solution.