Solutions to Exercises 2. Physics 603. (Nov 4, 2020)

1. Use the Bohr-Sommerfeld quantization condition that the orbit have a circumference containing an integral number of de Broglie wavelengths to find the allowed orbits of a 2D electron moving in a uniform magnetic field. Show that each successive orbit encloses precisely one additional quantum of flux in its interior. Hint: note the difference between the canonical momentum, which determines the de Broglie wavelength, and the dynamical momentum which determins the velocity. Use the symmetric gauge $\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B}$ in which the vector potential is purely azimuthal and constant in magnitude around the orbit.

An electron in a cyclotron orbit of radius r, has de Broglie wavelength λ determined by $n\lambda = 2\pi r$, so that the canonical momentum is given by

$$p = \frac{h}{\lambda} = n\frac{\hbar}{r} \tag{1}$$

For a vector potential $\vec{A} = \frac{\vec{B} \times \vec{r}}{2}$, the mechanical momentum is

$$\Pi = mv = p + eA = \left(\frac{n\hbar}{r} + \frac{eBr}{2}\right) \tag{2}$$

so that the total kinetic energy is given by

$$E(r) = \frac{(p+eA)^2}{2m} = \frac{1}{2m}\left(\frac{n\hbar}{r} + \frac{eBr}{2}\right)^2 \tag{3}$$

Differentiating with respect to r to obtain the minimum energy, we find

$$\frac{\partial E}{\partial r} = \frac{1}{m}\left(\frac{n\hbar}{r} + \frac{eBr}{2}\right)\left(-\frac{n\hbar}{r^2} + \frac{eB}{2}\right) = 0 \tag{4}$$

so that

$$\frac{\hbar n}{r^2} = \frac{eB}{2} \Rightarrow \Phi = \pi r^2 B = n\left(\frac{h}{e}\right) = n\Phi_0 \tag{5}$$

2. Express the exact lowest-Landau-level two-body eigenstate

$$\psi(z_1, z_2) = (z_1 - z_2)^3 e^{-\frac{1}{4}(|z_1|^2+|z_2|^2)} \tag{6}$$

as a sum of all possible two-body Slater determinants.

The two body eigenstate can be written

$$\Psi(z_1, z_2) = (z_1 - z_2)^3 e^{-\frac{1}{4}(|z_1|^2+|z_2|^2)} = [(z_1^3 - z_2^3) - 3(z_1^2 z_2 - z_2^2 z_1)]e^{-\frac{1}{4}(|z_1|^2+|z_2|^2)} \tag{7}$$

so we can write this as a sum of two Slater determinants as follows

$$\begin{aligned}\Psi(z_1, z_2) &= \left(-\begin{vmatrix} 1 & 1 \\ z_1^3 & z_2^3 \end{vmatrix} + 3\begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix}\right) e^{-\frac{1}{4}(|z_1|^2+|z_2|^2)} \\ &\equiv -\Psi_{3,3}(z_1, z_2) + 3\Psi_{21}(z_1, z_2) \end{aligned} \tag{8}$$

where $\Psi_{m,m'}$ corresponds to the Slater determinant with a particle in angular momentum states with azimuthal quantum number m and m'.

3. In quantum mechanics, for any single particle wavefunction $\psi(x)$, the particle has an uncertainty in its position x and momentum p set by the uncertainty principle, $\Delta x \Delta p \geq \frac{\hbar}{2}$. Show that for a wave function $\psi(x, y)$ in the nth Landau level, there is a similar uncertainty relation $\Delta x \Delta y \geq (n + 1)l^2$, where $l = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length. (Hint: write x and y in terms of the guiding center and mechanical momentum, and use this to evaluate an expression for $\langle x^2 \rangle$ and $\langle y^2 \rangle$, from which you can obtain a lower bound for $\Delta x \Delta y$.)

Let us write the x and y co-ordinate in terms of guiding center co-ordinates as follows

$$\begin{aligned} x &= R_x - \frac{l^2}{\hbar}\Pi_y \\ y &= R_y + \frac{l^2}{\hbar}\Pi_x. \end{aligned} \tag{9}$$

It then follows that

$$x + iy = R_x + iR_y + i\frac{l^2}{\hbar}(\Pi_x + i\Pi_y) = \sqrt{2}l(b^\dagger + ia) \tag{10}$$

where

$$a = \frac{l}{\sqrt{2}\hbar}\left(\Pi_x + i\Pi_y\right), \qquad b = \frac{1}{\sqrt{2}l}(R_x - iR_y). \tag{11}$$

are canonical bosons, so that $x^2 + y^2 = 2l^2(b^\dagger + ia)(b - ia^\dagger)$. If we evaluate $x^2 + y^2$ in a state $\Psi\rangle$with n filled Landau levels, then since $\langle\Psi|b^\dagger a^\dagger|\Psi\rangle = \Psi|ba|\Psi\rangle = 0$, it follows that

$$\langle\Psi|x^2 + y^2|\Psi = 2l^2\langle n_a + n_b + 1\rangle \tag{12}$$

We can without loss of generality take a state located at the origin, with $\langle n_b\rangle = 0$, in which case $\langle x^2 + y^2\rangle = \langle\Delta x^2 + \Delta y^2\rangle$, so that

$$\langle\Psi|\Delta x^2 + \Delta y^2|\Psi \geq 2l^2\langle n_a + 1\rangle = 2l^2(n + 1) \tag{13}$$

This is as far as we can go legally with this problem. If we take the state to be rotationally invariant, then $\Delta x^2 = \Delta y^2 = l^2(n+1)$, and we can write

$$\Delta x \Delta y \geq l^2(n+1). \tag{14}$$

4. A system of charge e Bosons in a partially filled Landau level at a high magnetic field is observed to develop quantum Hall plateaux.

(a) Write down the Laughlin wavefunction for a $\nu = 1/(2p)$ filled Landau level composed of hard core bosons. What is the expected Hall constant?

(b) Adapt Jain's idea of composite fermions to this case: consider each boson in the partially filled Landau level to be bound to an odd number $2p+1$ flux tubes, forming a composite fermion. Suppose the composite fermions occupy the first q Landau levels, calculate the filling factor ν of the original bosons, to predict the main sequences of fractional Hall constant values ($\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2}$) for hard core bosons.

(c) We can also think of the bosonic Laughlin state as a superfluid in which each boson carries $2p$ flux tubes. Write down the Chern Simons Ginzburg Landau action that describes this situation. What value of k is required to attach the $2p$ flux tubes?

(d) How might an experiment be carried out to confirm your predictions?

(a) The Laughlin wavefunction for a $\nu = 1/(2p)$ th filled Landau level of hard-core bosons is

$$\psi_m = \prod_{i>j}(z_i - z_j)^m \exp\left[-\frac{1}{4}\sum_j |z_j|^2\right], \qquad (m = 2p). \tag{15}$$

(b) We can rewrite the Laughlin wavefunction for bosons as

$$\psi = \prod_{i'>j'}(z_{i'} - z_{j'})^{2p-1}\prod_{i>j}(z_i - z_j)\exp\left[-\frac{1}{4}\sum_j |z_j|^2\right], \tag{16}$$

which we can interpret as a filled Landau Level of fermions, where each fermion is carries to $2p-1$ flux tubes. Since the statistical phase of $2p-1$ flux tubes is $\theta = (2p-1)\pi$, the combination of fermion and $2p-1$ flux tubes is a boson. But the boson is the microscopic particle, so we may equivalently regard the fermion as a composite of a boson with $2p-1$ flux tubes in the opposite direction to the field, i.e a composite fermion. Let ν be the filling factor of the bosons, and $N_\Phi = BA/\Phi_0$ to be the number of magnetic flux tubes, then the effective number of flux tubes seen by the composite fermions is

$$N'_\Phi = N_\Phi \mp (2p-1)N \tag{17}$$

depending on whether the attached fluxes are antiparallel or parallel to the external field. Now $N_\Phi = N/\nu$, whereas $N'_\Phi = N/q$, so that

$$\frac{N}{q} = \frac{N}{\nu} \mp (2p-1)N \tag{18}$$

or

$$\frac{1}{q} = \frac{1}{\nu} \mp (2p-1) \tag{19}$$

and hence

$$|\nu| = \frac{q}{(2p-1)q \pm 1}. \tag{20}$$

The Laughlin sequence is obtained by putting $q = 1$ and taking the plus sign, which gives $\nu = 1/2,\ 1/4,\ 1/6$. The Jain sequences, obtained by taking $p = 1$, and varying q are

$$\nu = \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \dots \tag{21}$$

and

$$\nu = 2, \frac{3}{2}, \frac{4}{3} \dots \tag{22}$$

with corresponding Hall constant $\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2}$.

(c) This is really the same as we discussed in class,

$$\begin{aligned} S \; &= \int d^2x \left\{ \psi^*(x)[iD_0 + \mu]\psi(x) - \frac{1}{2M}|D\psi|^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\eta}a_\mu \delta_\nu a_\eta \right\} \\ &+ \; S_{Coulomb} + S_{local}, \end{aligned} \tag{23}$$

excepting now, m is even, i.e $k = 1/m = 1/2p$.

(d) This last question was intended to get you thinking. One can't use $He-4$, as this is neutral. One possibility is to use cold atoms in an optical trap, in which one can create synthetic gauge fields, either by spinning the potential and using the Coriolis force, or by using Raman to create a synthetic vector potential. See Lin et al, Synthetic magnetic fields for ultracold neutral atoms, Nature 462 628 (2009).