Due date: Monday, April 24, at the end of the lecture.

Note: you are allowed to use only the textbook (Grosso & Parravicini), lecture notes, materials that I specifically provided, and Mathematica or other similar software to do the homework.

1. **Conductivity of a superconductor in the two-fluid model.** In the two-fluid model for a superconductor, it is assumed that there exist both normal and superconducting electrons. The normal electrons obey the Drude-like equation

\[ \frac{d\vec{j}_n}{dt} = \frac{n_ne^2}{m}\vec{E} - \frac{\vec{j}}{\tau}, \]

where \( n_n \) and \( \vec{j}_n \) are the number and current densities of the normal electrons, respectively. The superconducting electrons obey the London equation

\[ \frac{d\vec{j}_s}{dt} = \frac{n_se^2}{m}\vec{E}, \]

where \( n_s \) and \( \vec{j}_s \) are the number and current densities of the superconducting electrons, respectively.

1. Find the (total) frequency-dependent complex conductivity \( \sigma(\omega) \) for a superconductor. Use time dependence of the form \( e^{-i\omega t} \) for time-dependent quantities and assume that the normal and superconducting fluids respond independently to the electric field.

2. Show that, in the low-frequency limit, the response of the normal fluid is purely ohmic, while the response of the superconducting fluid is purely inductive.

2. **Magnetic field inside an infinite superconducting plate.** Solve the London equations for an infinite superconducting plate of finite thickness \( 2t \). Assume that the magnetic field of magnitude \( B_0 \) is applied parallel to the plate. Find both the magnetic field and the supercurrent inside the plate. Plot the magnetic field and supercurrent for \( 2t = \lambda_L \) and \( 2\lambda_L \).

3. **Critical field of a type-I superconductor in the Ginzburg-Landau theory.** Find the temperature dependence \( H_c(T) \) of the critical filed of a type-I superconductor for \( T \) close to \( T_c \) within the Ginzburg-Landau theory.

4. **Proximity effect between two planar superconductors.** Two planar superconductors 1 and 2 are placed with their flat faces in very good contact. Their critical temperatures are \( T_{c1} \) and \( T_{c2} \), respectively, with \( T_{c2} > T_{c1} \) and \( T_{c2} - T_{c1} \ll T_{c1} \). The system is cooled to a temperature \( T \) between \( T_{c1} \) and \( T_{c2} \), so that only superconductor 2 is superconducting.
1. Show that the Ginzburg-Landau equation for superconductor 1 can be written as

\[-\xi_1^2 \frac{d^2 \phi}{dx^2} + \phi + \phi^3 = 0,\]

where $\phi$ is the dimensionless wave function, and determine $\xi_1$.

2. Making use of the fact that $|\phi| \ll 1$ in a normal metal so that the cubic term in the above equation can be neglected, show that the wave function decays according to $\phi = \phi_0 e^{-|x|/\xi_1}$ in superconductor 2, where $x = 0$ is at the interface between the two superconductors and superconductor 2 occupies the $x < 0$ region.

5. *Finite-momentum BCS state.* The finite-momentum BCS state is given by

\[|\Psi_S(K)\rangle = \prod_K (\sum_{\sigma} u_k c_{k+K/2}^\dagger c_{-k+K/2}^\dagger)|0\rangle,\]

where $u_k$ and $v_k$ are the same as for $K = 0$.

1. Compute the order parameter $\Delta_K = g \sum_k \langle c_{-k+K/2} c_{k+K/2} \rangle$ in $|\Psi_S(K)\rangle$ (in terms of the energy gap $\Delta_0$ for the zero-momentum BCS state).

2. Evaluate the current density carried by the finite momentum BCS state, i.e. the expectation value of the current density operator

\[\hat{j}(q) = -\frac{e\hbar}{mV} \sum_{k\sigma} (k + q/2) c_{bfk\sigma}^\dagger c_{k+q,\sigma},\]

where $V$ is the system volume, in $|\Psi_S(K)\rangle$. What is the current density in the zero momentum BCS state we used in class?