Let's place our metal into an a.c. electric field or what if irradiate the metal by light (e.g. laser).

We start with the same eqm.

The problem can be solved in the same Drude model where \( \mathbf{\Phi} = -e \mathbf{E}(t) = -e \Re \{ \mathbf{E}(\omega) e^{-i\omega t} \} \)

so \( \frac{d\mathbf{\Phi}}{dt} = -i\omega \mathbf{\Phi} - e \mathbf{E}(t) \) we will seek a steady state solution i.e.

\[
\mathbf{\Phi} = \frac{e}{i\omega + \frac{1}{\tau}} \mathbf{E}(\omega) e^{-i\omega t}
\]

Since \( \mathbf{\Phi} = -\frac{ne}{m} \mathbf{P} \Rightarrow \mathbf{P}(t) = \Re \{ \mathbf{\Phi}(\omega) e^{-i\omega t} \} \)

\[
\mathbf{P}(t) = \Re \{ \mathbf{E}(\omega) e^{-i\omega t} \}
\]

\[
\mathbf{P}(\omega) = \frac{e}{\frac{1}{\tau} - i\omega} \mathbf{E}(\omega)
\]

\[
\Re \{ \mathbf{P}(\omega) \} = \frac{e}{\frac{1}{\tau} - i\omega} \frac{\mathbf{E}(\omega)}{i\omega}
\]

Recall \( j = \sigma \mathbf{E} \Rightarrow \sigma(\omega) = \frac{ne^2}{m} \frac{\mathbf{E}(\omega)}{\frac{1}{\tau} - i\omega}
\]

\[
\sigma(\omega) \text{ is known as a.c. conductivity.}
\]

Note \( \omega \to 0 \to \sigma(\omega \to 0) = -\frac{ne^2}{m} \frac{1}{i\omega} = \frac{\sigma_0}{1 + \frac{1}{\tau} - i\omega} \)

\( \sigma_0 \) is anisotropic conductivity.

Note: We ignored the magnetic field term. If we include it \( -\frac{e}{mc} \mathbf{P} \times \mathbf{H} = \mathbf{E} \) by about \( \frac{\omega}{c} \) so at the 1st approximation we have
However the most important complication is that both $E$ and $H$ vary in space and time. But our $\sigma(\omega)$ was derived under the assumption that it is \textit{spatially uniform}.

A few points are due:

1) the value of $\bar{j}$ at $\bar{r}$ is determined by $E$ right after the very last collision, for any realistic material the mean free path as such that the last collision is few m.f.p. away from $F$. So the fixed $E(\bar{r})$ should not vary too much. That is why we can write down:

$$j(\bar{r}, \omega) = \sigma(\omega) E(\bar{r}, \omega), \quad \lambda \gg \text{m.f.p.}$$

so this all valid if $\lambda$ of the radiation $\gg$ mean free path

Typical values for $\lambda_s$:

- Red light (visible) $\approx 800$ nm
- Soft X-rays (400eV) $\approx 1$-2 nm
- Hard X-rays (10keV) $\approx 1$ Å

Now we can calculate the response function $e(\omega)$ of a metal subjected to a.c. field.

Recall $j(\omega) = \sigma(\omega) E(\omega)$ and we look $j \approx \omega e$

$$E(\bar{r}, \omega) = E(\bar{r}) e^{i\omega t}$$

\[\begin{align*}
\nabla \times H &= \frac{\iota \mu_0}{\epsilon_0} j + \frac{1}{\epsilon} \nabla \times E \\
\nabla \times E &= -\frac{\iota}{\epsilon} \nabla H \\
\n\Rightarrow -\nabla^2 E &= \frac{\iota^2 \mu_0}{\epsilon^2} j + \frac{1}{\epsilon^2} \nabla^2 E \\
\Rightarrow -\nabla^2 E &= \frac{\iota^2 \mu_0}{\epsilon^2} j + \frac{1}{\epsilon^2} \left( \frac{\iota^2 \mu_0}{\epsilon^2} j \right)
\end{align*}\]
\[ -\nabla^2 E = \frac{\omega^2}{\varepsilon^2} E \quad \text{where} \quad \varepsilon(\omega) = 1 + \frac{4\pi N}{\omega}
\]

if \( \omega \) is high, let's see what we get:

\[ \varepsilon(\omega) \approx \omega \gamma \gg 1 = 1 + \frac{4\pi N \hbar}{\omega} \approx \frac{\hbar e^2}{m} \approx \frac{1}{\imath \omega c}
\]

then

\[ \varepsilon(\omega) = \omega \gamma \gg 1 = 1 + \frac{4\pi N \hbar}{\omega} \approx \frac{\hbar e^2}{m} \approx \frac{1}{\imath \omega c}
\]

where

\[ \omega_p = \frac{\sqrt{4\pi N e^2}}{m}
\]

**Plasma Frequency**

If \( \omega < \omega_p \)

1) at \( \omega = \omega_p \), \( \varepsilon = 0 \) if \( \omega < \omega_p \)

\( \varepsilon \) is negative \( \Rightarrow \quad -\nabla^2 E = \frac{\omega^2}{\varepsilon^2} \varepsilon(\omega) E \)

\[ -\nabla^2 E = \frac{\omega^2}{\varepsilon^2} \varepsilon(\omega) E \]

**Negative**

2) However if \( \varepsilon > 0 \) \( \Rightarrow \omega > \omega_p \)

Recall \( \tau = \left( \frac{0.22}{\rho \mu} \right) \left( \frac{r_s}{a_0} \right)^3 \cdot \text{ns} \cdot \text{sec} \)

\[ \omega_p \tau = \frac{\sqrt{4\pi N e^2}}{m} \left( \frac{r_s}{a_0} \right) = 1.6 \cdot 10^3 \left( \frac{r_s}{a_0} \right)^{3/2} \left( \frac{1}{\mu_0 c} \right)
\]

\( \omega_p \tau \gg 1 \)

for any reasonable metal.
e.g. alkali metals like K, Na, Cs etc.

\[ V_p = \frac{2eU_p}{\pi} = 11.4 \cdot \left( \frac{r_\ast}{a_0} \right)^{-3/2} \cdot 10^{15} \text{ Hz} \Rightarrow \]

\[ \lambda_p \approx 2 \cdot 10^{-3} \text{ Å} \]

(2) Perhaps among strange things is the presence of "charge oscillations" in a metal.

From the continuity eqn.

\[ \nabla \cdot j = \frac{\partial p}{\partial t} \quad (\rho = \rho(c) e^{-i\omega t}) \]

\[ \begin{cases} \nabla \cdot j(c) = i \omega \rho(c) \\ \nabla \cdot E(c) = \gamma \pi \rho(c) \end{cases} \quad \text{Gauss's law} \]

and \( \nabla \cdot j = -i \omega e \cdot \nabla \rho(c) \) \( \Rightarrow \) \( \nabla \cdot j = \nabla \cdot E(c) \)

\[ i \omega \rho(c) = (\omega \gamma \pi \rho(c) \rho(c) \left( 1 - \frac{\gamma \pi i \omega \rho(c)}{\omega} \right)) \Rightarrow \]

\[ \rho(c) \neq 0 \Rightarrow \]

The condition on the frequency should meet if the charge density wave is to propagate.

Here we introduce a new concept - excitations. The charge density wave before it gets quantized is also known as plasmons.

A classical point of view:

\[ \delta \text{ lets displace the ions} \]

\[ \frac{\epsilon}{\delta} = \frac{\epsilon_{\text{vac}}}{\delta} \]

\[ \text{hence} \quad \delta \text{ is not conductivity!} \]

its surface charge density.

For \( N \) electrons:

\[ N \vec{d} = -\frac{N e \times \mathbf{1}_{\text{vac}}}{\epsilon(\vec{r}, \vec{c})} = -\frac{N e \times \frac{1}{2} \mathbf{1}_{\text{vac}}}{\epsilon(\vec{r}, \vec{c})} \]

\[ \text{This leads to the oscillating motion.} \]
thermal conductivity in a metal

In addition to the theory of conductivity, Drude managed to explain what happens to a metal when we heat it up.

- Especially impressive how it works w.r.t.
  the famous Wiedemann and Franz law:

\[ \frac{K}{\sigma} = \alpha T \]

where \( \alpha \) is almost the same for all metals.

In other words

\[ \frac{K}{\sigma T} = \text{const}, \text{ e.g. } Li \frac{K}{\sigma T} = 2.22 \times 10^{-8} \text{ Watt}^{-1} \text{C}^{-1} \text{K}^2 \]

at 273K

Again as for the electrical conductivity

\( \frac{K}{\sigma T} = \text{const} \), e.g.

\( \text{Zn} = 2.28 \ldots \)

\( \text{Pb} = 2.64 \)

\( \text{Al} = 2.14 \ldots \)

let's assume that thermal transport is driven by electrons as well. Moreover, he suggested that since metals are more conductive than metals, the role of electrons must be very important compared to ions.

\[ \frac{\partial T}{\partial t} = -\sigma \mathbf{J} \]

\( \nabla T = \frac{\partial T}{\partial x} \) is the direction of heat flow.

\[ \mathbf{J} = \frac{\partial T}{\partial x} \]

\( \nabla T \) is the thermal gradient.

\( \text{thermal conductivity} > 0 \)

b/c thermal current is opposite to the thermal gradient.
lets assume we have a long thin rod.

\[ \frac{\partial T}{\partial x} = -k \frac{dT}{dx} \] 

Now after each collision, emerges with the velocity \( n + \) the local temperature, so interestingly even if \( \langle U \rangle = 0 \) the electron arriving from the hotter side will have higher energy than from the lower. So the net thermal flow is still perfectly maintained.

\[ \frac{\delta}{\delta x} E_1 - E_2 = \frac{\delta}{\delta x} \langle VT \rangle \cdot (r_n - r_i) = \frac{C_v}{N} \langle VT \rangle (r_n - r_i) \]

V.S. 

\[ \text{Hot} \quad \text{T} \]

\[ \text{Low} \]

This is the amount of energy transferred from \( r_1 \) to \( r_2 \) in a homogeneous system \( \langle U_n \rangle = \langle U_2 \rangle \), so \( \langle U_n \rangle = \frac{3}{2} k_B T \). For 1 electron, higher T.

From kinetic theory of gases

\[ \lambda = \frac{1}{3} \frac{C_v}{N} \frac{e^2}{h} \]

where \( \lambda = \frac{\langle U \rangle}{N} \) root mean square velocity \( \langle U \rangle \) is the average velocity, \( C_v \) is the heat capacity at constant V, per volume.

\[ C_v = \frac{3}{2} \frac{k_B}{h} \]

\[ \frac{h}{e} = \frac{1}{3} \frac{C_v}{N} \frac{e^2}{h} \]

\[ \frac{h}{m} e^2 = \frac{k_B}{e^2} \]

we can calculate \( \langle U \rangle \) from \( N \cdot \langle U \rangle = \frac{3}{2} k_B T \). \( u = \frac{3}{2} k_B T \)

\[ \langle U \rangle = N \]

\[ \frac{1}{3} \frac{C_v}{N} \]

Which is great BUT only \( \frac{1}{2} \) of the experimental value.

Note: in the original paper Drake made an error of \( \frac{1}{2} \) so his result was perfect.
derivation of \( K \) in the acoustic theory of gases.

So we have determined 
\[
E(r_2) - E(r_1) = \frac{c}{N} \frac{2T}{3} \Delta r \cdot \frac{x}{x^2} - \frac{5x^2}{2}.
\]

So how many electrons within time \( \Delta t \) go from \( r_1 \) to \( r_2 \)

The number of electrons going through this tunnel:

So the energy transfer:

\[
Q = -\frac{c}{N} \frac{2T}{3} A \cdot n \Delta t = -\frac{c}{N} \frac{2T}{3} A \cdot n \left< u_x^2 \right> \Delta t.
\]

So \( Q = -\frac{c}{N} \frac{2T}{3} A \cdot n \left< u_x^2 \right> \Delta t \)

The thermal current \( j = \frac{Q}{A \Delta t} = -\frac{c}{N} \frac{2T}{3} \nabla \cdot \left< u_x^2 \right> \nabla T = -\frac{c}{N} \frac{2T}{3} \nabla \Delta T = \)

For all 3 directions.

\( \frac{c}{V} \left< u_x^2 \right> \frac{2T}{3} \nabla T \)

\( \frac{c}{V} \) heat capacity per volume.

Often we can approximate \( \sqrt{\left< u_x^2 \right>} = \sigma \) thus

\( j = -\frac{c}{N} \frac{2T}{3} \nabla \sigma \), if we define the m.f.p

\( \sigma = \frac{c}{V} \)

\( \nu = -\frac{c}{N} \frac{2T}{3} \nabla \sigma \) so \( K = \frac{c}{N} \frac{2T}{3} \sigma \)
Consider another interesting effect:

\[ T_1, \Delta T, T_2 \]

\[ \text{DT flow of electrons} \]

will produce an internal electric field in the opposite direction to the flow of electrons.

This field is known as the thermoelectric field or the Seebeck effect.

\[ E = \varphi \Delta T \]

thermopower.

To estimate the thermopower, we calculate electronic velocity at the point \( x \) (1D case)

mean \( v_0 = \frac{1}{2} [v(x - \Delta x) - v(x + \Delta x)] = -\Delta x \frac{dv}{dx} = -\frac{d}{dx} \left( \frac{v^2}{2} \right) \]

To generalize to 3D \( \langle v_x^2 \rangle = \frac{1}{3} v^2 \) so that

\[ \rightarrow v_0 = -\frac{e}{6} \frac{d}{dT} \cdot \Delta T \]

recalling the average velocity due to the field gradient

\[ \overline{v_x} = -\frac{e}{6} \overline{E} \]

To have \( v_0 + v_e = 0 \)

\[ \frac{d}{dT} \frac{d^2}{dt^2} = -\frac{e}{m} \cdot \overline{E} \Rightarrow \overline{E} = \frac{m}{e} \frac{d}{dT} \cdot \Delta T \]

or \( \varphi = -\frac{m}{6e} \frac{dU^2}{dT} = -\frac{1}{2} \frac{d}{dt} \left( \frac{du^2}{dt} \right) \frac{dE}{dt} = -\frac{1}{2} \frac{d}{dt} \left( \frac{dE}{dt} \right) \]

Drude made another "mistake" stating

\[ \frac{C}{U} = \frac{3}{2} k_B \]

so \( \varphi = -\frac{k_B}{2e} = -0.12 \times 10^{-3} \mu V/k \)

But for metals at \( kT \)

the thermopower \( \varphi (\text{Li}) \sim \mu V/k \) or 100 times less!

This is a failure of the Drude theory.

We need quantum mechanics to account for these kinds of strange results!

THE END of L7