Drude theory of metals

Basics of Drude theory of metals.

- The theory of metallic conductivity - 1900. Electric and thermal

- Drude knew about electrons as negative particles inside solids from J.J. Thomson. To compensate the negative charge there should be positive particles in a metal and immobile.

\[ e = 4.8 \times 10^{-10} \text{ esu} = 1.6 \times 10^{-19} \text{ C} \]

\[ \text{e}^+ \text{electrons are detached} \]

\[ \text{e}^- \text{electrons are} \]

\[ \text{conduction electrons} \]

\[ \text{core electrons} -Z + Z^+ \]

\[ Z^+ - Z = e (Z^+ - Z) \]

Drude treated the conduction electrons as a gas of non-interacting "atoms" on the electrostatic background of anionic core.

Fast forward to more modern times.

There are \( 6.02 \times 10^{23} \) atoms/mole = \( N_A \). If we know density \( \rho = \frac{\text{mass}}{V} \) we get \( \frac{\rho}{N_A} = \frac{\text{mass}}{\text{atOMIC weight} \cdot \text{Vol}} \) if each atom contributes \( Z \) electrons. Total # of electrons:\n
\[ \frac{N}{V} = N_A \cdot Z \frac{\rho}{A} = \text{carrier density} = n \]
\( E_1 = 6.10^{-23} \ \text{eV} \cdot 0.5 \ \text{g/cm}^3 = \frac{6.5 \times 10^{-4} \ \text{eV}}{A/2} \approx 5.10^{-22} \ \text{eV/cm}^3 \) \\
\( h = 27.7 \) \ , \( \pi = 17.0 \) \( h_c = 0.91 \)

Drude introduced electron density:

\[ \frac{1}{n} = \frac{V}{N} = \frac{4\pi}{3} r_s^3 \Rightarrow r_s = \left( \frac{3}{4\pi} \cdot \frac{1}{n} \right) \]

\( r_s \sim 10^{-8} \ \text{cm} = 10^{-8} \cdot 10^{-2} \ \mu = 10^{-10} \ \mu = 1 \ \text{Å} \)

we can scale \( r_s \) to the radius of H atom:

\( a_0 = \frac{k^2}{\mu^2} = 0.529 \cdot 10^{-8} \ \text{cm} = 0.529 \cdot 10^{-10} \ \mu = 0.529 \ \text{Å} \)

e.g. in many cases \( r_s/a_0 \sim 2 \ \text{or} \ \frac{1}{3} \)

despite e-e and e-ion interactions they are ignored.

\( 1 \) no external B and E electron moves straight

\( 2 \) if external field then electron moves according to Newton's law without interactions with other e-

\( 3 \) the role of collisions is to instantly change the velocity and direction of an electron.

Drude thought of bouncing of the ionic cores.

known to day as the independent electron approximation.
Suprisingly, e-e interaction in normal metals is NOT important.

- Collisions happen with the probability of $1/e$ (in units of time).
  More physically, it means an electron will travel on average $e$ units of time between the collisions.

  \[ \tau \equiv \text{the relaxation time} = \text{the collision time} = \text{the mean free time} \]

5) Electrons lose their energy via collisions.

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**Theory of dc electric conductivity**

According to the Ohm's law, $V = IR$.

Resistance $R$ is a function of wire size.

Can we explain this?

To avoid issues with the wire size, let's introduce $\rho = \text{resistivity}$ such as

\[ \vec{E} = \rho \vec{J} \]

is amount of charge per second per crossing area.

So $J = \frac{I}{A}$ and $V = E \cdot L$.
from $E = \rho j$, $j = \frac{I}{A}$

$$\frac{V}{I} = \frac{\rho}{A} = R$$

More microscopically, if $n$ electrons move with $\bar{v}$, $\nu d\bar{x}$ is the distance they cover in $dt$, so $n\nu (\text{volt})A$ will cross the volume $dV$, the total charge crossing this volume: $-e n (dt\nu)A$

The current density then $\overrightarrow{J} = -e n \bar{v}$ [e$^-$/sec.cm$^2$]

$\bar{v}$ is the average velocity which is $\approx 0$, no net electric charge. If $\bar{v} \neq 0 \Rightarrow \bar{v}$ is not 0

b/c $\bar{v} \neq 0$!

Let's move to even more microscopic view.

$t$ is the time since last collision

$\bar{v}_0$ - after the collision

$\bar{v}_0 - \frac{eE}{m}t \Rightarrow$ let's average this over many collisions

$$\bar{v}_{\text{avr}} = \langle \bar{v}_0 \rangle - \langle \frac{eE}{m}t \rangle$$

$$= 0$$

$$\bar{v}_{\text{avr}} = -\frac{eE}{m}\bar{v}$$

and since

$$\overrightarrow{J} = -en\bar{v} = -en (\frac{eE}{m}\bar{v}) = \left(\frac{ne^2e}{m}\right)\overrightarrow{E}$$

Thus $\sigma \bar{v} = \text{conductivity} = \frac{ne^2e}{m}$

$$\overrightarrow{J} = \frac{\sigma}{\rho} \overrightarrow{E} \Rightarrow \overrightarrow{J} = \frac{1}{\sigma} \overrightarrow{E} \Rightarrow j = 6.2 \overrightarrow{E}$$

Now: we know $n$ from the simple estimates $n = N_A \frac{2p}{A}$

$\frac{e^2}{m}$ are constants, what is not known is $n$.

That's why often we use transport measurements to quantify $\eta = \frac{6m}{n \epsilon^2}$
Typical numbers,

Li $\rho (\mu S/cm)$ $\rho (KT/77k)$
RT 77k 12.4 1.04 $\sim 12$

Ag 2.94 0.5 $\sim 5.6$

Bi 156 35 $\sim 5$

We can express $\tau$ in terms of $\tau$ by using $r_s$

$$\tau = \left( \frac{0.22}{\rho \mu} \right) \left( \frac{r_s}{a_0} \right)^3 \times 10^{-14} \text{ sec}$$

From this we can conclude that for typical materials $\tau \sim 10^{-14} 10^{-15} = 10^{-14} \text{ fm sec at RT}$

Mean free path then can be expressed as:

$$\ell = \frac{\tau}{\sigma} = \frac{m\sigma_0^2}{2} = \frac{3}{k_B T} \Rightarrow \sigma_0 = \sqrt{\frac{k_B \ell}{m} \ell} \sim 10^{-7} \text{ cm}$$

at RT

$$\ell_{mp} \sim 10^{-7} \times 10^{-14} = 10^{-21} \text{ cm} \approx 1 \text{ Å}$$

So, since $1 \text{ Å} \approx a = \text{lattice size}$ may be it's true that $e^-$ collides with the positive ions.

Since we don't have a theory of $\tau$ (by the way experimentally it's measured very precisely by angular resolved photoemission ARPES) we focus on quantities which are $\tau$ independent!

**Case 1:** Metals and dielectrics in d.c. field

**Case 2:** $\sim \mu$ in a.c. field.
Consider case 1: \( \overrightarrow{v} = \overrightarrow{P}(t)/m \) \( \Rightarrow \frac{d}{dt} = -en\overrightarrow{v} = -en\overrightarrow{P}(t)/m \)

Since we know \( p \) at \( t \), what's \( p \) at \( t + dt \)?

Now, electron scatters within time \( dt \) with probability \( \frac{dt}{\tau} \)

and not scattering \( 1 - \frac{dt}{\tau} \)

\[ \frac{dt}{\tau} \]

\[ 1 - \frac{dt}{\tau} \]

Contribution of those not scattered \( e^- \) is

\[ p(t) + f(t) \, dt \]

average mom. extra work due \( f \)

\[ \overrightarrow{P}(t + dt) \approx \left( 1 - \frac{dt}{\tau} \right) \left[ \overrightarrow{P}(t) + f(t) \right] = p(t) - \frac{dt}{\tau} \overrightarrow{P}(t) + f(t) \, dt \]

the correction to this sign from scattered carriers is \( \approx O(dt)^2 \), i.e. scattered electrons contribute only this fraction \( \frac{dt}{\tau} \)

we ignore the scattered electrons

\[ p(t+dt) - p(t) = -\frac{dt}{\tau} p(t) + f(t) \, dt \]

\[ \frac{\overrightarrow{P}(t+dt) - \overrightarrow{P}(t)}{dt} = \frac{df}{dt} = -\overrightarrow{P}(t) + f(t) \]

the effect of the external force is to dump the momentum \( \overrightarrow{P}(t) \) by \( e \).
\[ wc = \frac{eH}{mc} \]

\[ o = -eEx - wcpy - px/c \]

\[ o = -eEy + wcpx - py/c \]

Recall \( j = -ne\vec{v} \)

\[ j \cdot m = -ne\mu \vec{v} = -ne\vec{P} \Rightarrow p = \frac{j \cdot m}{en} \]

so

\[ px = -\frac{Jx \cdot m}{en} \]

\[ py = -\frac{Jy \cdot m}{en} \]

\[ o = -eEx + \frac{wc \cdot Jx}{en} \cdot jy + \frac{Jx \cdot m}{en} \]

\[ o = -eEy - \frac{wc \cdot Jy}{en} \cdot jx + \frac{Jy \cdot m}{en} \]

Also recall: \[ \sigma = \frac{ne^2c^2}{m} \]

\[ \sigma_0 Ex = wc \cdot jy + jx \]

\[ \sigma_0 Ey = -wc \cdot jx + jy \]

In the steady state \( jy = 0 \) ! \Rightarrow \[ \{ \begin{align*}
\sigma_0 Ex &= jx \quad \text{kind of trivial} \\
\sigma_0 Ey &= -wc \cdot jx \Rightarrow Ey = -\frac{wc}{\sigma_0} jx
\end{align*} \]

And from \( wc = \frac{eH}{mc} \)

\[ \sigma = -\frac{e^2n^2c^2}{m} \]

\[ \Rightarrow \quad E_y = -\left( \frac{H}{nec} \right) jx \quad \Rightarrow \]

\[ RH = \frac{E_y}{jxH} \Rightarrow RH = -\frac{H}{nec} \frac{jx}{jxH} = \frac{1}{nec} \]

\[ RH = -\frac{1}{nec} \]
Oddly enough, $R_H$ depends on $H, T$, etc.

Why? It turned out the value of $R_H$ where $n$ is calculated from the valence electron participating in conduction is only valid for ultra-pure elemental metals at very low $T$ and high magnetic field $H$.

Another comment: $w_C$ can be a good measure of the external magnetic field.

i.e., when $w_C$ is small, e.g., $6 \times E_x = \frac{\omega_C}{\omega} j_x + j_x$

$6 \times E_x = \frac{\omega_C}{\omega} j_x + j_y$

so $j$ and $E$ are almost parallel to each other. However, when we increase the angle between $E_x$ and $E_y$ gets larger.

Hall angle $= \tan \theta = \frac{E_y}{E_x} = \frac{J_x H}{n e c E_x} = \frac{6 E_x H}{n e c E_x} = \frac{K e^x c H}{n e c m} = -\omega_C$

i.e., long relaxation time implies large Hall angle $\theta$.

For experiment: a good estimate of $w_C$:

$w_C (GHz) = 2.8 \times H (kB), \quad w_C = \frac{\omega_C}{2 \pi}$
Hall effect.

A non-magnetic material placed into a magnetic field, when we attempt run current across it:

\[ \mathbf{F}_L = -e \mathbf{v} \times \mathbf{B} \]

In the equilibrium \( \mathbf{F}_L = F_{\text{Hall}} \) due to the surface build up at the edge of the sample or surface.

To quantify this phenomenon Hall introduced 2 ratios:

\[ \frac{E_x}{J_x} = \rho(H) = \text{magneto-resistance} \]

Hall coeff. \( \rho_H = \frac{E_y}{J_x H} \)

Since \( E_y \) is negative, \( \rho_H \) is negative too.

**RULE:** For electrons \( \rho_H > 0 \)

holes \( \rho_H < 0 \)

by measuring the sign of the Hall coeff. one can tell what kind of carriers are dominant at the Fermi edge.

To calculate \( \rho(H) \) and \( \rho_H \), we can go back to:

\[ \frac{dp}{dt} = -e \left( E + \frac{p}{m_e} x H \right) - \frac{p}{\tau} \]

\[ \frac{dp_x}{dt} = -(eE_x + wc p_y) - p_x/\tau \]

\[ \frac{dp_y}{dt} = -(eE_y + wc p_x) - p_y/\tau \]

in the steady condition

\[ \frac{dp}{dt} = 0 \]

END