Lecture 5

Fluctuations in the 2nd order parameter, critical indices and scaling, quantum phase transitions.

Close to $T_c$ fluctuations become really important and this leads to modifications in $c_p$, $\kappa$, compressibility $\kappa$, etc. For example, instead of a jump in $c_p$ there will be a real singularity, i.e.,

$$C(T) \sim \ln \left( \frac{1T - T_c}{T_c} \right) \sim (T - T_c)^{-\alpha}$$

So the real issue is where exactly the fluctuations are really important?

answer: $\langle \Delta y^2 \rangle \propto y^2$

From Ginzburg-Levanyuk theory

$$\xi \sim \frac{1T - T_c}{T_c} \sim \frac{B^2}{\frac{8\pi^2 \kappa^4 T_c^2}{\xi_0^6}} \equiv \frac{B^2 T_c}{8\pi^2 \kappa^4 \xi_0^6}$$

where $\xi_0 = \sqrt{\frac{\kappa}{\alpha T_c}}$ - is known as the correlation length at zero $T$.

def: The correlation length $\xi$ defines a typical scale in the ordered state $T_\infty$ the equilibrium order parameter "returns" to the equilibrium value when "disturbed".

Inside $\xi$ fluctuations are so important that they can modify all the thermodynamic quantities.
To quantify this we introduce the Ginzburg number $\xi^2_T \equiv \frac{B^2 T_c}{8\pi^2 a}\xi_c^3$

and if $\xi^2_T << 1$, we can use Landau theory without restrictions, i.e.

$$\xi^2 \sim \frac{1}{\xi_c^2} = \frac{\xi^2_T}{\xi_c^2} = \xi^2_T << 1 \Rightarrow \xi_c$$ is large

For example for superconductors

$\xi_0 \sim 10^{-9} \text{ m}$ - the so-called coherence length is $\gg a$, the lattice parameter.

But for $^3\text{He}$ $\xi_0 \sim a$ and the fluctuations are always super strong.

\[ \text{The } \Lambda\text{-curve or the } \Lambda\text{-anomaly.} \]

An explanation: $b/c$ phase fluctuations are very strong, above $T_c$ the phase doesn't disappear right away, instead the phase still fluctuates. Above $T_c$ we can call this phenomenon as short-ordering. This in turn means the whole entropy is not released at $T_c$ and some remains above $T_c$. Also close to $T_c$ fluctuations destroy the expected behaviour for $C$ (when $T \to T_c$)

In general: The spatial size of the region where fluctuations are important are defined by the exchange interaction.
e.g., if the interaction range of a long range, the bi cell, the fluctuations are small.

A meaning of $\xi$: Consider fluctuations of the order parameter at $\mathbf{0}$ and $\mathbf{r}$, the are correlated

$$<\Delta \eta (0) \cdot \Delta \eta (r)> \sim \frac{T}{t} e^{-r/\xi}$$

and $\xi (T) = \sqrt{\frac{6}{\pi (T-T_c)}}$, note $\xi_0 = \xi (T=0)$

within $\Gamma$ b/c of the fluctuations there are regions of phase fluctuation.

We can also describe the fluctuations in momentum space, e.g.

$$<\Delta \eta \cdot \Delta \eta - \eta> = <|\eta|^2> \Rightarrow$$

$$\frac{1}{4\pi} \int \frac{e^{-r/\xi}}{r} e^{-ikr} d^3 r = \frac{2\pi}{\frac{1}{2}} \int_0^\infty e^{-r/\xi} e^{-ikr} r^2$$

$$\sin \theta \, dr \, d\theta \, d\phi = \frac{2\pi}{\frac{1}{2}} \int_0^\infty e^{-r/\xi} e^{-ikr} r^2$$

in polar coordinates.

$$\sin \theta \, dr \, d\theta \, d\phi = \text{let } u = \cos \theta \Rightarrow du = -(\sin \theta) \, d\theta$$

$$= -2\pi \int_1^1 \int_0^\infty e^{-r/\xi} e^{-ikr} \, dr \, du = 2\pi \int_1^1 \int_0^\infty e^{-r/\xi} e^{-ikr} \, dr =$$

integrate over $u$ first

$$= 2\pi \int_0^\infty r \, e^{-r/\xi} \left[ -\frac{1}{i kr} e^{-ikr} + \frac{1}{i kr} \right] dr =$$

$$= 2\pi \int_0^\infty r \, e^{-r/\xi} \int_{-1}^1 e^{-ikr} dr =$$
\[ T \left( \frac{1}{k} \right) = \frac{2\pi}{ik} \int_0^\infty e^{-r/\beta} \left[ e^{-ikr} + e^{ikr} \right] dr = \]
\[ = \frac{2\pi}{ik} \int_0^\infty \left[ e^{-\left(ik + \frac{1}{3}\right)r} + e^{\left(ik - \frac{1}{3}\right)r} \right] dr = \]
\[ = \frac{2\pi}{ik} \left[ \frac{e^{-\left(ik + \frac{1}{3}\right)r}}{\frac{1}{3} + ik} + \frac{e^{\left(ik - \frac{1}{3}\right)r}}{\frac{1}{3} - ik} \right]_0^\infty = \]
\[ = \frac{2\pi}{ik} \int \left[ \frac{e^{-\left(ik + \frac{1}{3}\right)r}}{\frac{1}{3} + ik} - \frac{e^{\left(ik - \frac{1}{3}\right)r}}{\frac{1}{3} - ik} \right]_0^\infty = \]
\[ = \frac{2\pi}{ik} \int \left( \frac{1}{3} - ik \right)e^{-\left(ik + \frac{1}{3}\right)r} - \frac{1}{3} + ik \right)e^{\left(ik - \frac{1}{3}\right)r} = \]
\[ = \frac{2\pi}{ik} \int \left[ \frac{1}{3} e^{-\left(ik + \frac{1}{3}\right)r} - ike^{-\left(ik + \frac{1}{3}\right)r} - \frac{1}{3} e^{\left(ik - \frac{1}{3}\right)r} - ike^{\left(ik - \frac{1}{3}\right)r} \right]_0^\infty = \]
\[ = \frac{2\pi}{ik} \left[ \frac{-\frac{1}{3} + ik + \frac{1}{3} + ik}{k^2 + \frac{1}{3} \lambda} \right] = \frac{2\pi}{ik} \frac{2ik}{k^2 + \frac{1}{3} \lambda} = \frac{4\pi\cdot i}{k^2 + \frac{1}{3} \lambda} \]
\[ = \frac{4\pi}{3\sqrt{k^2 + \frac{1}{3} \lambda}} \cdot T \quad \text{So the answer is} \]
\[ \langle \delta x \delta y \rangle^2 \sim \frac{4\pi T \frac{3\lambda^2}{1 + \frac{1}{3} \lambda^2}}{1 + \frac{1}{3} \lambda^2} \sim \frac{4\pi T}{1 + \frac{1}{3} \lambda^2} \]

This is a famous Ornstein–Zernike theory of fluctuations.
There are interesting relationships between $\langle \eta(\mathbf{q}) \rangle^2$ and the response of a system to an external perturbation.

In fact, we are familiar with some of these: $\varepsilon(\mathbf{q}, \omega)$ and $\chi(\mathbf{q}, \omega)$

$\uparrow$ dielectric constant $\uparrow$ magnetic susceptibility.

For static susceptibility $\chi(\mathbf{q}) = \langle \eta(\mathbf{q}) \eta(0) \rangle$

$$\chi(\mathbf{q}) = \frac{1}{T} \int \frac{d^3 \mathbf{r}}{(2\pi)^3} e^{i \mathbf{q} \cdot \mathbf{r}} \langle \eta(\mathbf{r}, \omega) \eta(\mathbf{r} + \mathbf{q}) \rangle$$

which is exactly what the Fourier transformation is about.

$p. 4$ \(\frac{1}{\sin \theta} = \frac{4 \pi T \frac{g^2}{1 + q^2 \xi^2}}{1 + q^2 \xi^2 (T)}

\text{Close to } T_c

\frac{\chi(\mathbf{q})}{T > T_c} = \frac{4 \pi \xi^2 (T)}{1 + q^2 \xi^2 (T)}

where $\xi(T) = \sqrt{\frac{\hbar}{4 \pi a (T - T_c)}}$

and for the static measurements in a SQUID magnetometer $\mathbf{q} = 0$, we get

$$\chi(0) = \frac{4 \pi G}{q (T - T_c)} \text{ or for } T > T_c$$

\[
\chi(0) \sim \frac{1}{a (T - T_c)}
\]