Lecture 4

So far we only considered the case when \( \gamma(r) = \text{const.} \).

But what if \( \gamma = \gamma(r) \) or we would say it's \textit{inhomogeneous}?

In this case the total energy is:

\[
\Phi = \int d^3r \left( \text{free energy density } \Phi(r) \right)
\]

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No! The variation of the order parameter in space cost energy!

Slow variation seems the very best approach, i.e. \( \frac{du}{dr} \sim \nabla \gamma(r) \).

The only space invariant which contains \( \nabla \) is \( \nabla^2 \), i.e. \( (\nabla \gamma)^2 \) would go into the free energy term.

By direct analogy, the Gibbs free energy can be written as:

\[
\Phi^* = \int d^3r \left\{ A \gamma^2(r) + B \gamma^4(r) + C (\nabla \gamma)^2 \right\}
\]

This is known as \textit{Ginzburg-Landau functional} or \textit{Ginzburg-Landau-Wilson functional}.

This form of the \textit{functional} is super popular and is used in the theory of \textit{superconductivity}, theory of \textit{domain walls} etc.
Few words about superconductivity:

If we minimize this fun. w.r.t \( \psi(\mathbf{r}) \) or \( \psi \) we get something new and not the ordinary self-consistent term \( \frac{d\Phi}{dy} = 0 \).

Instead we are going to get the famous Landau-Lifshitz equation:

\[
-\frac{i}{2m} \left( -i\hbar \nabla - \frac{2e}{c} \mathbf{A} \right)^2 \psi + \lambda \psi + eB|\psi|^2 \psi = 0
\]

here \( \mathbf{A} \) - is the vector potential.

This eq. is very similar to the Sch. eqn. but with the extra non-linear term \( \lambda |\psi|^2 \).

Now close to \( T_{sc} \) where \( |\psi| \) is very small, we can linearize this equation, and solve exactly like the Sch. eqn.

The very same equation can be applied to the theory of ferromagnets and is known as Landau-Lifshitz eqn. to describe domains and domain walls, etc.

Recall for the homogeneous solution if \( g \) coeff. \( g_0 \) then we get a usual 2nd order phase transition.

But what if \( g (g_0, T) \leq 0 \)? in this case there should be a transition from the homogeneous solution to the inhomogeneous one!
\[\Phi = \int d\bar{v}^3 \left\{ A \bar{v}^2 + B \bar{v}^4 + C (\nabla \bar{v})^2 + E (\bar{v}^2)^2 \right\}\]

To make our life simpler let's move into momentum space e.g. \((\bar{v}^2)^2 \rightarrow q^2\)

\[\Phi = \int \ldots \left( q^2 \right)^2 + E \left( q^2 \right)^2\]

\[\frac{d\Phi}{dq^2} = 6q^2 + 2Eq^2 \frac{q^2}{2E} = 0\]

\[\Rightarrow \text{min}_{q^2} = \Phi = \frac{C}{2E}(6\alpha)\]
This means that a new structure with a period \( \frac{2\pi}{q} = L \) will be formed. Notice that \( L \) doesn't have to be commensurate with the lattice period \( \tilde{a} \).

The point \( P^*, T^* \) where \( G(P^*, T^*) \) changes its sign is called the Lifshitz point. Here is a typical phase diagram for the system with this kind of transition:

- **Disorder**
- **Homogeneous phase**
- **Inhomogeneous phase**
- **Lifshitz point**
- **First order phase transition**
- **Control parameter**, e.g. chemical doping, pressure, etc.