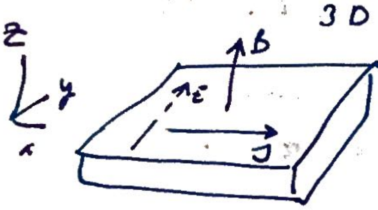
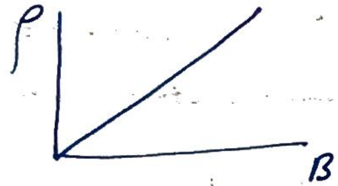
Quantum Hall effect and Topological insulators

Here is a very interesting observation.

3D crystal

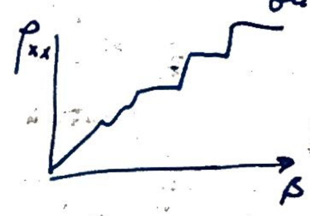


Semiconductor e.g. GaAs



2D electron liquid

2 semicond. interfaces e.g. GaAs + GaAs :Al



$\rho_{xx} = E_x / j_x$

$\rho_{xy} = E_y / j_x$

Based on the cylindrical symmetry

$$\rho_{xx} = \rho_{yy} \qquad \rho_{xy} = \rho_{yx}$$

From classical mechanics (Debye theory) $\rho_{xy} \sim B$

$$eE_y = evB \qquad j = evn \Rightarrow \rho_{xy} = \frac{E_y}{J_x} = \frac{B}{en}$$

Very useful but nothing too spectacular:

Quantum Hall effect and its connection to quantum electrodynamics

At low T and very very clean samples the 2DEG don't follow $\rho_{xy} \sim B$!

Instead is shows a series of very strange plateaus. On the platoe $\sigma_{xy} = \frac{1}{\rho_{xy}}$ is quantizes $= \frac{e^2}{h} n$ $\quad n = 1, 2, 3, \ldots$ and $\rho_{xx} = 0$!

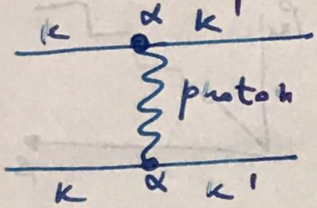
the filling factor.

Those platos are known as the Q. H. E.

Interestingly the value of conductivity can be expressed in terms of the fine structure constant:

$$\alpha = \frac{e^2}{\hbar c} \qquad \sigma_{xy} = n \frac{e^2}{\hbar} \Rightarrow = n \cdot \alpha \cdot c$$

The f.s. constant α measures the strength of quantum electrodynamics:



now $e^2 \sim$ coulomb potential
denominator $\sim c$

$$\alpha \sim \frac{\text{potential energy}}{\text{kinetic energy}}$$

If in our universe $\alpha = 0$ then electrons will not interact and in fact there will be <u>no photons</u> (no light - just imagine this).

If α is large the universe will be made of very <u>strongly entangled matter</u> which will make the presence of life impossible as we know it.

However $\alpha \sim \frac{1}{137}$ which is just 1% of the kinetic energy and we leave in the world with interactions $\equiv$ photons $\equiv$ light and yet we can do the perturbation theory.

Consider just kinetic energy as the unperturbed term, then in the 1st approx

$O(\alpha) \sim 1\%$, $O(\alpha^4) \sim 10^{-8}$!

Amazing but a condensed matter experiment can be as accurate as high energy physics in defining α!

QED:

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

CMP: $\alpha_{CMP} = \dfrac{e^2}{\hbar v_{Fermi}}$ for a typical solid $v_F \sim \frac{1}{100} - \frac{1}{1000} c$

So $\alpha_{CMP} = \dfrac{e^2}{\hbar v_F} \sim 1 \div 10$

so the perturbation theory don't work.

QHE and Topology.

Before we answer why σ_{xy} is quantized Lets try to think why $\sigma_{xx} = 0$.

Q: if $\rho_{xx} = 0$ is it a superconductor / or "perfect Metal

A: NOoooo.. The material is ~~SC~~ – ~~Conductor~~ – Insulator! wow....

The material has 0 conductivity.

$j = \sigma E$, $E = \rho j$ $\rho = \frac{1}{\sigma}$

This is only true if j and E are in the same direction. However more generally:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \cdots & \sigma_{xz} \\ \vdots & \ddots & \vdots \\ \sigma_{zx} & \cdots & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

if all but $\sigma_{xx}\, \sigma_{yy}\, \sigma_{zz}$ are not zero

$j = \sigma E$ and $\rho = \frac{1}{\sigma}$ but for 2D

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$ and the resistivity

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}^{-1} =$$

$$= \frac{1}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \begin{pmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{pmatrix}$$

Now lets go to to the platoe where $\rho_{xx} = \rho_{yy} = 0$

$$\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \rho_{xy} \\ \rho_{yx} & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} = \frac{1}{-\rho_{xy}\rho_{yx}} \begin{pmatrix} 0 & -\rho_{xy} \\ -\rho_{yx} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/\rho_{yx} \\ 1/\rho_{xy} & 0 \end{pmatrix} \Rightarrow \sigma_{xx} = \sigma_{yy} = 0$$ NO CONDUCTIVITY so the system is INSULATOR.

But between the platoes the conductivity is non zero and as such it's METAL. $\rho_{xx} \neq 0$ and σ_{xx} and $\sigma_{yy} \neq 0$.

SUMMARY OF EXP. FACTS ABOUT QHE.

1. When we vary the external field
- we can turn system ... ins $\leftrightarrow$ metal $\leftrightarrow$ ins. $\leftrightarrow$ metal ...
- each insulating state corresponds to a plateau of ρ_{xy} and the step between 2 neighboring plateaus is metallic.
- transport for the metalic states is NOT universal, and changes from sample to sample.
- The insulating state is UNIVERSAL. $\rho_{xx} = 0$ and $\sigma_{xx} = 0$ and σ_{xy} is quantized.

• WHY IQH state is insulator?

LANDAU LEVELS.

Lets solve this ~~pb~~ problem quantum mechanically. In 2D for a charge neutral particle $q=0$. the Schrödinger eqn:

$$i\frac{\partial \psi(x,y)}{\partial t} = \left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y}\right)^2\right]\psi(x,y)$$

with $H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y}\right)^2$

Now lets include charge and its connection to E and B. He we will use a minimal coupling which tells that we change momentum $\bar{p} \to \bar{p} + \frac{e\bar{A}}{c}$ A = vector potential

and $i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} - \frac{e}{c}\phi$ ϕ = electric potential

$$\left(i\frac{\partial}{\partial t} - \frac{e}{c}\Phi\right)\Psi(x,y) = \left[\frac{1}{2m}\left(-i\hbar - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2\right]\Psi(x,y)$$

$$i\frac{\partial}{\partial t}\Psi(x,y) = \left\{\left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2\right] + \frac{e}{c}\Phi\right\}\Psi(x,y)$$

$$H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}A_y\right)^2 + \frac{e}{c}\Phi$$

Since $E=0$ we can set $\Phi = 0 \Rightarrow$

$$\nabla\times A = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = B$$

from E & M we know that A is not observable i.e. for the fixed B field, A is not uniquely defined: $\nabla\times A = B$ $A' = A + \nabla\chi$ is also the vector potential for the same field $\nabla\times A' = B$. We can choos any A as long as as $\nabla\times A = \nabla\times A'$. Now lets apply the magnetic field along z : B_z Then we can use 2 options: $A_x = 0$ $A_y = Bx$ (the Landau gage)

$A_x = -\frac{By}{2}$ and $A_y = \frac{Bx}{2}$ (the symmetric gage)

$$\left.\begin{array}{l}\frac{\partial}{\partial x}A_y = B\\ \frac{\partial}{\partial x}A_x = 0\end{array}\right\} \nabla\times A = B.$$

Energy spectrum

$$H = \left[\frac{\hbar^2}{2m} \left(-i\frac{\partial}{\partial x}\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{c}Bx\right)^2 \right]$$

Static Sch. equation: $H\Psi = E\Psi$

in the Landau gage $[p_y, H] = 0$ therefore we can find common eigen states for p_y and H.

$$\Psi(x,y) = f(x)\, e^{-ik_y y}$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{1}{2m}\left(\hbar k_y - \frac{e}{c}Bx\right)^2 f(x) = \epsilon f(x)$$

(for p_y the eigenvalue $\hbar k_y$), lets rewrite it:

$$-\frac{\hbar^2}{2m} f''(x) + \frac{e^2}{2mc^2}B^2\left(x - \frac{c\hbar}{eB}k_y\right)^2 f(x) = \epsilon f(x)$$

$$-\frac{\hbar^2}{2m} f''(x) + \frac{K}{2}(x - x_0)^2 f(x) = \epsilon f(x)$$

where $x_0 = \frac{c\hbar}{eB}k_y$ $\quad K = \frac{e^2B^2}{mc^2}$; $x_0 = \ell_c^2 k_y$ where $\ell_c^2 = \frac{c\hbar}{eB}$

now this equation looks like the harmonic oscillator:

Recall from your QM class, the solution is

$$\begin{cases} \Psi_{n,k_y}(x,y) = \phi_n(x - x_0)\, e^{-ik_y y} \quad \text{with} \\ \epsilon_{n,k_y} = \left(n + \frac{1}{2}\right)\hbar\omega = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{K}{m}} = \left(n+\frac{1}{2}\right)\underbrace{\frac{eB}{cm}}_{\omega_c}\hbar \end{cases}$$

$n = 0, 1, 2, 3 \ldots$

So the electrons are quantized in x-y and have continuous translation along z; so the total E

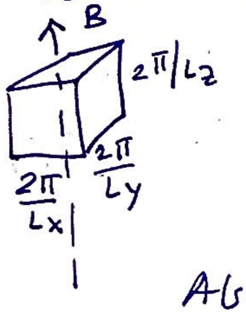
$$E = E(n, k_y) + \frac{\hbar^2}{2m}k_z^2 = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2}{2m}k_z^2$$

L16

If we include the spin of the electron then each level will split into 2 sublevels $\pm g_e \mu_B \sigma H$

Since the energy spectrum is dramatically affected we need to see what happens to the electronic density of states.

We start with



$\Rightarrow$? $\Rightarrow$ k_x and k_y are quantized in units $\frac{2\pi}{L_x}$ and $\frac{2\pi}{L_y}$

Also recall $x_0 = \frac{c\hbar}{eB} k_y = \frac{2\pi \ell_c^2}{L_y}$

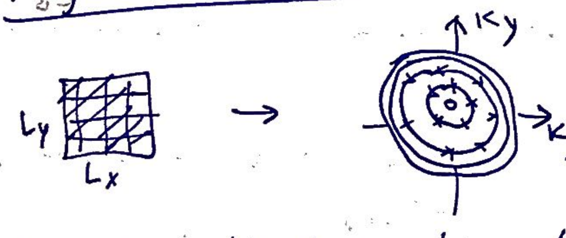
And the degeneracy of the level in 2D is:

$D = \frac{L_x}{\Delta x_0} = \frac{L_x L_y}{2\pi \ell_c^2}$; The total magnetic flux through the x-y plane is

$\Phi = H L_x L_y$ and the flux quantum: $\phi_0 = \frac{\hbar c}{e} \Rightarrow$

$D = \frac{\Phi}{\phi_0} \overset{or}{\Rightarrow}$ the number of states = number of the flux quanta in units of $\frac{\hbar c}{e}$!

The <u>Physical Meaning</u>:



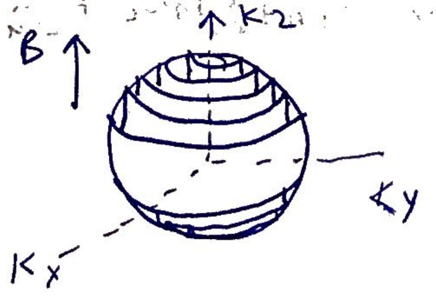
After applying the magnetic field points distributed in the $k_x - k_y$ area get spread out onto concentric circles with energies $\frac{\hbar\omega_c}{2}, \frac{3}{2}\hbar\omega_c$, $\frac{5}{2}\hbar\omega_c \ldots$

But the total number of state remain the same. To show this lets calculate the number of states per unit area per unit energy and no spin

$$g(E) = \frac{1}{L_x L_y} \frac{D}{\hbar\omega_c} = \frac{m}{2\pi\hbar^2}$$

= which is the same as $= g^{2D}(E)$ without magnetic field attached.

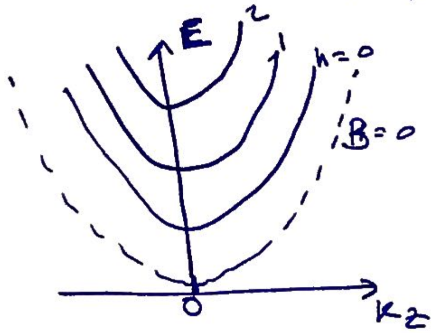
Now lets extend this to 3D:

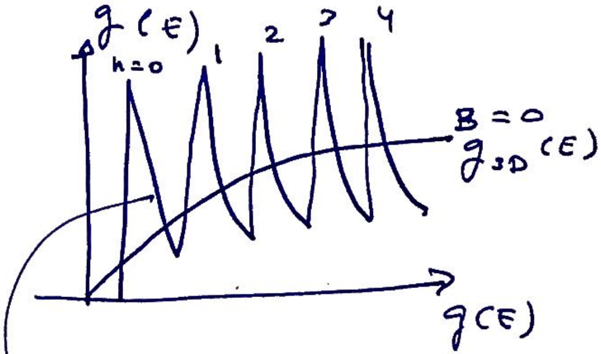


Note k_z is still a good quantum number so we can plot $E(k_z)$ vs. k_z as bands also known as the <u>Landau subbands</u>

Overall for 3D we have:

$$g_{3D}(E) = \frac{1}{(4\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_n \left[E - (n+\tfrac{1}{2})\hbar\omega_c\right]^{-1/2}$$

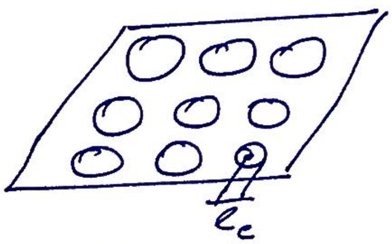


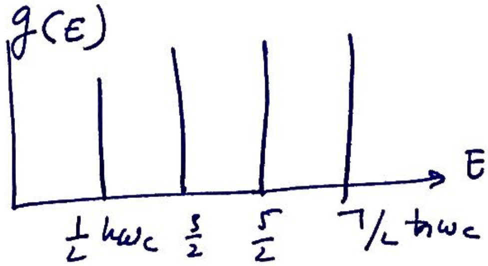


1D spikes from the states on the circumference of the circles.

$$g_{1D}(E) = \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^{1/2}(E-E_n)^{-1/2}$$

For 2D this is very interesting





If we apply E along x-direction we going to have the famous QHE.