Spin of Dirac Particle.

In the Heisenberg picture we get:

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

For a free Dirac particle $H = c\alpha \cdot p + \beta mc^2$

For this particle we introduce the time evolution of angular momentum L

$$-i\hbar\,(\,i\underbrace{\dot{L}_1}_{\equiv L_x} + j\underbrace{\dot{L}_2}_{\equiv L_y} + k\underbrace{\dot{L}_3}_{\equiv L_z}) = [H, iL_1 + jL_2 + kL_3]$$

lets consider one component at a time:

$$-i\hbar\frac{\partial L_3}{\partial t} = [H, L_3] = \Big[\sum_{k=1}^{3} c\,\alpha_k p_k + \beta mc^2, \underbrace{x_1 p_2 - x_2 p_1}_{\equiv L_3}\Big]$$

remember β is a matrix and $\Rightarrow$ $[\beta mc^2, x_1 p_2 - x_2 p_1] \equiv 0$!

$$-i\hbar\frac{\partial L_3}{\partial t} = \Big[\sum c\alpha_k p_k, x_1 p_2\Big] - \Big[\sum c\,\alpha_k p_k, x_2 p_1\Big]$$

$$= -i\hbar c\sum \alpha_k \delta_{k1} p_2 + i\hbar c\sum \alpha_k \delta_{k2} p_1$$

$$= -i\hbar c(\alpha_1 p_2 - \alpha_2 p_1) = -i\hbar(\alpha \times \bar{p})_3$$

$$\begin{cases} [p, x] = -i\hbar \\ [p_i\, p_j] = 0 \end{cases}$$

Similarly for L_2 and L_1

$$\frac{d\bar{L}}{dt} = c\,(\alpha \times \bar{p}) \neq 0$$

This means that L is not a constant of motion or conserved.

However we can construct a new operator such as to cancel out $-c(\alpha \times \bar{p})$, i.e.

$L + A$ = will be conserved if $\dfrac{dA}{dt} = -c(\alpha \times \bar{p})$!

Let us demonstrate that $A = \frac{\hbar}{2}\sigma'$ where

$$\sigma' = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

Lets calculate:

$$-i\hbar \frac{d\sigma_3'}{dt} = [H, \sigma_3'] =$$

$$= \left[\sum c\,\alpha_k p_k + \beta mc^2, \sigma_3'\right] = 2i\,(\alpha_1 p_2 - p_1\alpha_2)$$

again as above $[\beta mc^2, \sigma_3'] = 0$ $\quad [\alpha_3, \sigma_3'] = 0$

$\uparrow$ scalar $\quad \uparrow$ matrix

and $\sigma_1' \sigma_3' = -i\sigma_2'$ $\quad \sigma_1'\sigma_3' = -\sigma_3'\sigma_1'$

or for all the components:

$$\frac{d}{dt}\frac{\hbar}{2}\sigma_3' = -c\,(\alpha \times p)_3$$

etc

$$\frac{d}{dt}\frac{\hbar}{2}\sigma' = -c\,(\alpha \times \bar{p})$$

And hence for $\frac{d}{dt}\left(L + \frac{\hbar}{2}\sigma'\right) = + \dots - \dots = 0$

ORBITAL ANG. MOM. $\quad$ SPIN ANGULAR MOMENTUM

$L + \frac{1}{2}\hbar\sigma' = J \equiv$ total angular momentum.

and J is the conserved quantity.

Thus Dirac particle describes a fermion with Spin $= 1/2$

PARTICLES IN A POTENTIAL

In order to solve an interesting problem of the particle trying to penetrate the infinitely high barrier we 1st need to modify the equation to add a potential.

Intuitively $H = c\alpha\cdot\bar{p} + \beta mc^2 + V$

$$= -i\hbar c \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\alpha = \alpha_3} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc^2 + V$$

$$\left[-i\hbar c \frac{\partial}{\partial z} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + mc^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V\right] \begin{pmatrix} v \\ w \end{pmatrix} = E \begin{pmatrix} v \\ w \end{pmatrix}$$

by multiplying those matrices.

$$\begin{cases} -\hbar c \underbrace{w'}_{= \frac{dw}{dz}} + v(mc^2 + V) = Ev \\ \hbar c v' - w(mc^2 + V) = Ew \end{cases}$$

Recall $\psi_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$ and $\psi_2 = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$

with E_1 and E_2

$$\begin{array}{l} v_2 \times \\ w_2 \times \\ w_1 \times \\ v_1 \times \end{array} \begin{cases} -\hbar c w_1' + (mc^2 + V) v_1 = E_1 w_1 \\ \hbar c v_1' - (mc^2 + V) w_1 = E_1 v_1 \\ -\hbar c w_2' + (mc^2 + V) v_2 = E_2 v_2 \\ \hbar c v_2' - (mc^2 + V) w_2 = E_2 w_2 \end{cases}$$

$(-)$

$$\begin{cases} \hbar c (v_1 w_2' - w_1' v_2) = (E_1 - E_2) v_1 v_2 \\ \hbar c (v_1' w_2 - w_1 v_2') = (E_1 - E_2) w_1 w_2 \end{cases}$$

or

OR

$$\hbar c \frac{d}{dz} (v_1 w_2 - w_1 v_2) = (E_1 - E_2) \psi_1^* \psi_2$$

① if $E_1 = E_2$

$$\frac{d}{dz} (v_1 w_2 - w_1 v_2) = 0 \Rightarrow v_1 w_2 - w_1 v_2 = \text{const}$$

Since v and $w \to 0$ if $|z| \to \infty$ $\Rightarrow$ const $= 0$

or $\frac{v_1}{w_1} = \frac{v_2}{w_2}$

$$\frac{v_1'}{v_1} = \frac{(E + mc^2 + V)}{\hbar c}\frac{w_1}{v_1} = \frac{(E + mc^2 + V)}{\hbar c}\frac{w_2}{v_2} = \frac{v_2'}{v_2}$$

this also means that $\psi_1 \propto \psi_2$

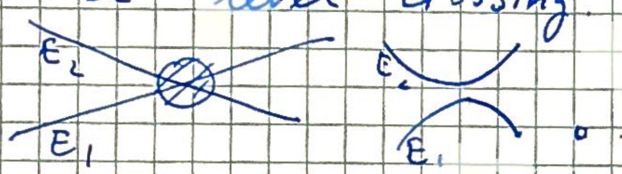
if $E_1 \neq E_2$ or the state is non-degenerate in the potential V

$$\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dz \qquad \int_0^{\infty} \left(v_1^* \; w_1^*\right)\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} dz$$

$$= \cancel{[illegible]} = \frac{1}{E_1 - E_2}\left(v_1 w_2 - w_1 v_2\right)\Big|_{-\infty}^{\infty} = 0$$

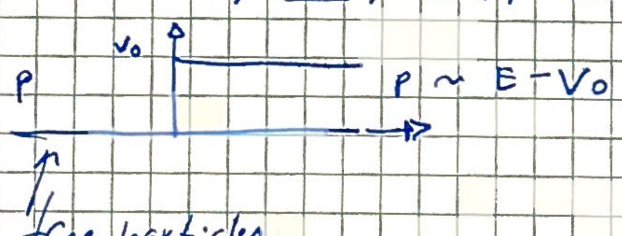
or ψ_1 and ψ_2 are orthogonal

This condition means that there cannot be level crossing.



KLEIN PARADOX

Scattering by Step Potential



$$v_<(x) = A\left(e^{ipx/\hbar} + R e^{-ipx/\hbar}\right)$$

Then $w_<(x) = A\left(a e^{ipx/\hbar} - a R e^{-ipx/\hbar}\right)$

where $a = \frac{cp}{E + mc^2}$

$$\psi_< = \begin{pmatrix} v_< \\ w_< \end{pmatrix} = A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} e^{ipx/\hbar} + R\begin{pmatrix} 1 \\ -a \end{pmatrix} e^{-ipx/\hbar}\right]$$

$$= A\left[u_+ e^{ipx/\hbar} + R u_- e^{-ipx/\hbar}\right]$$

where $u_\pm = \begin{pmatrix} 1 \\ \pm a \end{pmatrix}$. For $x>$ no reflected wave

$$\chi_> = \begin{pmatrix} v_> \\ w_{>0} \end{pmatrix} = D\,\bar{u}\, e^{ipx/\hbar}$$

where

$$\bar{u} = \begin{pmatrix} 1 \\ b \end{pmatrix}, \quad \bar{p} = \frac{1}{c}\sqrt{\left[(E-V_0)^2 - m^2c^4\right]}$$

$$b = \frac{c\bar{p}}{E - V_0 + mc^2}$$

To determine D:
$$\begin{cases} \Psi_<(0) = \Psi_>(0) \\ \Psi'_<(0) = \Psi'_>(0) \end{cases}$$

$$\begin{cases} D\,\bar{u}\, \cancel{e^{i\bar{p}x/\hbar}} = A\left[v_+ \cancel{e^{ipx/\hbar}} + R v_- e^{-ipx/\hbar}\right] \\ \text{- - - - -} \end{cases}$$

$$A(v_+ + R v_-) = D\bar{u} \Rightarrow$$

$$A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} + R\begin{pmatrix} 1 \\ -a \end{pmatrix}\right] = D\begin{pmatrix} 1 \\ b \end{pmatrix} \Rightarrow$$

$$\begin{cases} A(1+R) = D \\ Aa(1-R) = bD \end{cases} \Rightarrow R = \frac{a-b}{a+b} \quad T = \frac{2a}{a+b}.$$

1. Now consider $E > V_0 + mc^2$
2. $V_0 - mc^2 < E < V_0 + mc^2$
3. $E < V_0 - mc^2$

Case 1: $E > V_0 + mc^2$

$$E^2 > m^2c^4 + V_0^2 + 2mc^2V_0 \Rightarrow E^2 - m^2c^4 > 0$$

$$p^2c^2 = E^2 - m^2c^4 > 0 \Rightarrow p \text{ is real.}$$

also, $E - V_0 > mc^2$ $\quad (E-V_0)^2 > m^2c^4$ or

$$\bar{p} = \left((E-V_0^2)^2 - m^2c^4\right]^{1/2} > 0$$

$\Rightarrow \bar{p}$ is real.

This means for $\Psi_< = \ldots =$

$$A\left[u_+ e^{ipx/\hbar} + R u_- e^{-ipx/\hbar}\right]$$

↑ transmitted wave ← reflected wave

So this behavior is the same as for non-relativistic Sch. equation

Case 2 $V_0 - mc^2 < E < V_0 + mc^2$

$(E - V_0)^2 < m^2c^4 \Rightarrow \bar{p} = \frac{1}{c}\left[\underbrace{(E-V_0)^2}_{<0} - m^2c^4\right]^{1/2} \Rightarrow$ so $\bar{p}$ is imaginary!

Then: $\Psi_>(x) = A T \bar{u}_+ e^{-i\bar{p}x/\hbar}$

So the wave is exponentially decaying

In the region I we have right moving incoming and left moving reflected waves

Nothing special, very similar to NRQM.

Case ③ Now $E < V_0 - mc^2$

$E - V_0 < -mc^2 \Rightarrow (E - V_0)$ negative.

and $(E - V_0)^2 > m^2c^4 \Rightarrow \bar{p}$ is real.

~~leading to an oscillatory in II~~

this means oscillatory behavior in region II and for NRQM there is no such solution

Moreover $E - V_0 + mc^2 < 0 \Rightarrow b < 0$

Then $|R| = \left|\frac{a-b}{a+b}\right| = \left|\cancel{\frac{a+|b|}{a-|b|}}\right| > 1$

Klein PARADOX [THE AMPLITUDE OF THE REFLECTED WAVE IS LARGER THAN THE INCIDENT ONE !

More particles are refect than arrive to the barrier ?!]

Even for very small E compared to V_0 we still get oscillatory behavior inside region II [SEE page 485, sec 19.8.2 for solution]

L8

$$R = \frac{E+p}{E-p} \cdot \frac{(p+V_0-q)^2}{(p+V_0+q)^2}$$

$$T = \frac{2p}{E-p} \cdot \frac{(E-V_0-q)^2 - m^2}{(p+V_0+q)^2} \quad (c=1)$$

here p is a momentum in Region I
and I changed the notation to q for Region II

At the limit $V_0 \to \infty$

$$R = \frac{E-p}{E+p} \quad \text{but} \quad \boxed{T = \frac{2p}{E+p}} \;\; ???!!!$$

even at the limit of the infinite barrier we have a finite transmittion probability.

Weird ?