

LECTURE

Identical Particles

on the way to quantum field theory

Particles are identical when we cannot distinguish them based on any physical property.

→ Many-body physics

o pure quantum phenomena

In classical mechanics we can trace any particle individually. Since for quantum we deal with the wave like properties of amplitudes of probability we cannot distinguish them any longer.



So unless we consider the case when 2 particles are well separated, i.e. without the overlap we cannot talk any longer about the individual objects in QM.

We need new formalism. As usual the best way to deal with this situation is to consider symmetry for the ensemble of the quantum particles.

① Permutation symmetry

if we consider a system below the threshold for getting it into the excited state we have 4 - degrees of freedom. i.e.

\vec{p} , \vec{r} , $\vec{\sigma}$ and $\epsilon \Rightarrow$ helicity
 ↑
 spin

Consider a system which has N -particles

- 1 $\equiv \vec{p}_1, \vec{r}_1, \sigma_1$
- 2 $\equiv \vec{p}_2, \vec{r}_2, \sigma_2$
- etc.

Hamiltonian can be written as:

$$H(1, 2, 3, \dots, N) \text{ and so is } \Psi(1, 2, \dots, N)$$

Lets introduce a new operator P_{ij} = the permutation operator $i \leftrightarrow j$ with eigenvalue ω .

$$P_{ij} \Psi(1, 2, \dots, i, \dots, j, \dots, N) = \omega \Psi(1, 2, \dots, j, \dots, i, \dots, N)$$

do this twice

$$P_{ij}^2 \Psi(\dots, i, \dots, j, \dots) = \omega P_{ij} \Psi(\dots, j, \dots, i, \dots) = \omega^2 \Psi(\dots, i, \dots, j, \dots) \Rightarrow \omega^2 = 1 \text{ or } \omega = \pm 1$$

\uparrow real

then P_{ij} is a hermitian operator.

$$P_{ij}^2 = I \Rightarrow P_{ij} = P_{ij}^{-1} \text{ also you can show that } P_{ij} \text{ is unitary.}$$

i.e. $U U^{-1} = I$

$$P H P^{-1} = H \Rightarrow P H = H P \text{ and for}$$

any symmetric operator O we get $[P, O] = 0$

if $\Psi(1, \dots, N)$ is eigenstate of H

then $P H \Psi = P E \Psi = E P \Psi$ is also an eigenstate

THIS IS CALLED EXCHANGE DEGENERACY

also all observables are the same for Ψ and $P \Psi$

Now we have $N!$ exchange operators for N particles, there has to be at least one eigenfunction such as:

$$H \Psi = E \Psi \text{ AND } P_{ij} \Psi = \omega_{ij} \Psi$$

also one can show that $P_{ij} P_{ik} = P_{jk} P_{ij} = P_{ik} P_{jk}$
 see PROBLEM #1 ch. 18

This ~~is~~ also means

$$P_{ij} P_{ik} \Psi = w_{ij} w_{ik} \Psi \Rightarrow$$

$$\frac{1}{2} w_{ij} w_{ik} = \frac{1}{2} w_{jk} w_{ij} = w_{ik} w_{jk}$$

which gives $w_{ik} = w_{jk}$ ~~$w_{jk} = w_{ij} = w_{ik}$~~
 or $w_{ij} = w_{ik} = w_{jk}$

if $w_{ij} = 1$, then Ψ is symmetric under interchange of any pair of particles $\equiv \Psi_S(1 \dots N)$

if $w_{ij} = -1$ Ψ antisymmetric $\equiv \Psi_A(1 \dots N)$

$$P_{ij} \Psi_S(1 \dots i \dots j \dots N) = \Psi_S(j \dots i) = +1 \Psi_S(i \dots j)$$

$$P_{ij} \Psi_A(i \dots j) = - \Psi_A(i \dots j)$$

$$P \Psi_A = \Psi_S$$

$$P \Psi_A = (-1)^P \Psi_A \quad (-1)^P = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \end{cases} \text{ permutations}$$

also $P \Psi = e^{i\theta} \Psi$ where $\theta = 0, \pi, \pi$

if θ is any $0 < \theta < \pi$ the particle is called anyon

e.g. Consider two particles $N=2$

P_{12} is the only operator

for 3-particles $P_{ij} \Psi(1, 2, 3)$

$i=1, 2, 3$
 $j=1, 2, 3$

- P_{12}
- P_{13}
- P_{23}
- P_{23}

$$\left. \begin{aligned} P_{12} P_{13} \Psi &= \\ &= P_{12} \Psi(3, 2, 1) = \\ &= \Psi(3, 1, 2) \\ P_{13} P_{12} \Psi(1, 2, 3) &= \\ &= P_{13} \Psi(2, 1, 3) = \\ &= \Psi(2, 3, 1) \end{aligned} \right\}$$

do not commute!
 so there are states where all P_{ij} 's are not diagonal.

However, experimentally we know that only $\theta=0$ and $\theta=\pi$ are realized.

- So the Hilbert space is broken into subspaces H_s and H_a and the remaining unphysical $(N! - 2)$ states H_r .

Since P commutes with H the symmetric characters cannot be changed over time.
 s. fermions remain fermions \rightarrow bosons remain bosons.

- Moreover, if we have a system of identical particles they will be described by a uniquely symmetrical wave f , otherwise the states would mix between symm. and anti-symm.

Spin - statistics theorem.

- A wave function of N identical particles of $\frac{1}{2} \cdot n$ where n is odd must have antisymmetric wave function under exchange of any two particles.
- Wave function of any N particles with n where n is even or "0" must ~~be~~ be symmetric.

So there are 2 quantum statistics.

Bose - Einstein $n = \text{even or } 0 \Rightarrow$ bosons

Fermi - Dirac $n = \frac{1}{2} m \quad m \equiv \text{odd} \Rightarrow$ fermions

it works even for composite particles

e.g.

$${}^3\text{He} = 2 \text{ protons} + \text{neutron} \\ 2 \cdot \frac{1}{2} + \frac{1}{2} = 1\frac{1}{2} = \text{fermion}$$

${}^4\text{He} = \text{boson} =$ and the subject to B-E condensation = superfluidity

Symmetry of w.f.

Many-body Sch. eqn.

$$i\hbar \frac{\partial}{\partial t} \Psi(1 \dots N) = H(1 \dots N) \Psi(1 \dots N)$$

Among all possible solutions we need to construct at least either symm. or antisymm. w.f.

Let's see how it's done.

Recall among $N!$ solutions corresponding the same energy eigenvalue $P\Psi$. So we can sum them up and then normalize.

This can be understood that if we have all bunch of $\Psi(1 \dots N)$

one of them into another which is includes into this sum. $P\Psi \rightarrow$ will just transform

how to { So the antisymm. can be setup as a sum of all ~~the~~ permuted w.f. by means of even interchanges of pairs and subtractions constructed the sum of all the functions by means of odd number of interchanges.

Example: 2 particles: ~~1, 2~~ Lets assume

$\Psi(12)$ is the solution of Sch. eqn.

then $P_{12} \Psi(12) = \Psi(21)$ is also a solution

$$\begin{aligned} \Psi_s &= A (\Psi(12) + \Psi(21)) & \Psi_a &= B (\Psi(12) - \Psi(21)) \\ P\Psi_s &= A (\Psi(21) + \Psi(12)) = \Psi_s & P\Psi_a &= B (\Psi(21) - \Psi(12)) \\ & & &= -B \Psi_a \end{aligned}$$

~~$$\Psi_{\text{tot}} = A \left(\Psi_1(x_1) + \Psi_2(x_2) \right)$$

$$= A \left(\Psi_1(x_2) + \Psi_2(x_1) \right)$$~~

For N particles

$$\Psi_s = S_s \Psi(1 \dots N, t) = \frac{1}{\sqrt{N!}} \sum_P \Psi(1 \dots N, t)$$

$$\Psi_a = S_a \Psi(1 \dots N, t) = \frac{1}{\sqrt{N!}} \sum_P (-1)^P \Psi(1 \dots N, t)$$

Ex. 3 particles: $\Psi(123)$ $N! = 3! = 1 \cdot 2 \cdot 3 = 6$

$$\Psi_s = \frac{1}{\sqrt{6}} \left[\Psi(123) + \Psi(213) + \Psi(132) + \Psi(321) + \Psi(312) + \Psi(231) \right]$$

$$\Psi_a = \frac{1}{\sqrt{6}} \left[\Psi(123) - \Psi(213) + \Psi(231) - \Psi(132) - \Psi(321) + \Psi(312) \right]$$

$P=1 \Rightarrow (-1)^1$ $P=2 \Rightarrow (-1)^2$ and so on.

Note for $\Psi_s^* \Psi_s$ and $\Psi_a^* \Psi_a$ the w.f. does not change under permutation of 2 particles. So any measurable quantity is not sensitive to the particle exchange.

$$|\Psi_a|^2 = \frac{1}{2} \left(\Psi(1,2)^2 + \Psi(2,1)^2 - \underbrace{\Psi(1,2)\Psi(2,1)}_{\text{interaction term!}} \right)$$

if particles are well separated this term = 0 and particles are distinguished.
 e.g. for $\epsilon \gg k_B T$ and low number of particles # of particles per quantum state is small and we can apply classical statistics.

Pauli Exclusion Principle

! No two fermions can be in the same quantum state (i.e. have the same quantum numbers).

It's not possible to solve a many body problem exactly.

The way we solve 1) assume particles are non-interacting, 2) include interactions via perturbation theory.

Specifically:

$$\begin{aligned}
 H(1 \dots N) &= H(1) + \dots + H(N) \Rightarrow \\
 \left\{ \begin{aligned}
 \Psi(1 \dots N) &= \Psi(1) \Psi(2) \dots \Psi(N) \\
 E_0 &= E_{\alpha_1} + E_{\alpha_2} + \dots + E_{\alpha_N} \\
 H_0(j) \phi_{\alpha}(j) &= E_{\alpha} \phi_{\alpha}(j)
 \end{aligned} \right.
 \end{aligned}$$

The eigenfunction corresponding to E_0 will be a linear combination of $\Psi(1 \dots N)$

In general Ψ_{α} can be written as determinant (Slater determinant)

$$\Psi_{\alpha} = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_{\alpha_1}(1) & \phi_{\alpha_2}(2) & \dots & \phi_{\alpha_N}(N) \\ \phi_{\alpha_2}(1) & & & \vdots \\ \vdots & & & \vdots \\ \phi_{\alpha_N}(1) & \phi_{\alpha_N}(2) & \dots & \phi_{\alpha_N}(N) \end{vmatrix}$$

$\phi_{\alpha_j}(j)$ with α_j wave functions.
 \uparrow j th particle

Change of sign comes from the change of sign upon exchange of 2 columns.

By looking at the Slater matrix
we can see also if any of two particles
say 1 and 2 have the same $\alpha_{12} \Rightarrow$
the determinant = 0

However for bosons, any two ~~or~~ or
more particles can occupy the same state
i.e. the occupation number for bosons 0, 1, 2 ...
for fermion it is either 0 or 1.

SPIN OF 2 ELECTRONS

$$H = \frac{1}{2m} \sum_{i=1}^N p_i^2 + V(r_1, r_2, \dots, r_N)$$

has no spin operator or if so is neglected.

$\rightarrow \Psi(r_1, \dots, r_N)$. IF we include spin
we need to add the spin eigenfunction
 $\chi(r_1, \dots, r_N)$ so

$$\Psi(1 \dots N) = \phi(r_1, \dots, r_N) \chi(r_1, \dots, r_N)$$

or some sort of linear combo of such
product $\phi \cdot \chi$. This is the 1st approx
for SO interaction.
 \uparrow
spin-orbit

We still need to apply the same symmetry
argument to the total wave function, ~~but~~
but now we consider both χ and ϕ .

The issue is that we can construct asym. and sym. parts for X and ϕ separately.

$$\left. \begin{array}{l} \text{sym} \times \text{sym} \\ \text{asym} \times \text{asym} \end{array} \right\} = \text{sym} \quad \leftarrow \text{only for bosons}$$

$$\text{sym} \times \text{asym} = \text{asym.} \quad \leftarrow \text{only possible for fermions}$$

Lets recall how it works for spins

For angular moment

$$\begin{array}{l} \vec{J}_1 \quad |j_1, m_1\rangle \\ \vec{J}_2 \quad |j_2, m_2\rangle \end{array} \Rightarrow \vec{J} = \vec{J}_1 + \vec{J}_2$$

$$|\psi\rangle = |j_1, m_1\rangle |j_2, m_2\rangle$$

$(2j_1 + 1)(2j_2 + 1)$ eigenstates total

The same can be applied to spins:

$$s = \frac{1}{2} \quad \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \alpha \quad \text{and} \quad \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \beta$$

The spin part for 2 electrons, 1 and 2

$$\alpha(1) \alpha(2)$$

$$\alpha(1) \beta(2)$$

$$\beta(1) \alpha(2)$$

$$\beta(1) \beta(2)$$

Next we can construct 4 commuting operators out of S :

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$$

$$S_z = S_{1z} + S_{2z}$$

$$S_{1z}^2$$

$$S_{2z}^2$$

the common set of eigenstates we label as $|S, m_S\rangle$ where

$$s = \frac{1}{2} + \frac{1}{2} = 1$$

$$m = -1, 0, 1$$

$$\frac{1}{2} - \frac{1}{2} = 0$$

$$\begin{Bmatrix} 10 \ 0 \ 0 \\ 11 \ -1 \ 7 \\ 11 \ 0 \ 7 \\ 11 \ -1 \ 7 \end{Bmatrix}$$

= 4 eigenkets in this coupled representation

The Clebsch-Gordan matrix for this representation is:

See Ch 11.12 of the text about 2 moments.

$$\begin{cases} \chi_s^{(1)} = |11\rangle = \alpha(1)\alpha(2) \\ \chi_s^{(0)} = |10\rangle = \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) + \beta(1)\alpha(2)] \\ \chi_s^{(-1)} = |1-1\rangle = \beta(1)\beta(2) \\ \chi_a = |100\rangle = \frac{1}{\sqrt{2}} [\alpha(1)\beta(2) - \beta(1)\alpha(2)] \end{cases}$$

$$\begin{cases} S^2 = (S_1 + S_2)^2 \cdot |S m_s\rangle = S(S+1) |S m_s\rangle \\ S_z |S m_s\rangle = m_s |S m_s\rangle \end{cases}$$

↑
in units of \hbar

The eigenstates for S_z are $\begin{cases} \chi_s^{(1)} = \hbar \\ \chi_s^{(0)} = 0 \\ \chi_s^{(-1)} = -\hbar \\ \chi_a = 0 \end{cases}$ S^2 eigenvalue = $2\hbar^2$

$\chi_s^{(1)}, \chi_s^{(0)}, \chi_s^{(-1)}$ = TRIPLET STATE AND SYMMETRIC
 χ_a = ANTI-SYMM.

Remember this is only the spin part.

The total wave function ~~is~~ $\chi \cdot \phi(r)$ still need to be ANTI-SYMM.

[SEE PROBLEM 3]
 very instructive page 453.

Exchange Interaction.

In non relativistic version of Q.M. we have no notion that interaction may depend on spin.

Consider a system of 2 electrons.

$$H(1,2) = K_1(r_1) + K_2(r_2) + V(r_2 - r_1)$$

in the c.m.
representation

$$\frac{1}{M} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2} = \frac{1}{\mu}$$

$$\begin{aligned} \vec{R} &= \vec{r}_1 + \vec{r}_2 \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}$$

$$H(r, R) = H_{rel}^{(1)} + K_{cm}(R) = \frac{1}{2\mu} P^2(r) + V(r) + K_{cm}(R) = \frac{1}{2M} P_{cm}^2(R)$$

$$P_{12} \text{ commutes with } H \Rightarrow [P_{12}, K_{cm}] = 0 = [P_{12}, H_{rel}] = 0$$

The eigenstates of K_{cm} and H_{rel} can be either symmetric or antisymm.

e.g. for K_{cm} : $e^{iK_{cm}R} = e^{iK_{cm}(r_1+r_2)}$

~~antisymm~~ $P_{12} e^{iK_{cm}(r_1+r_2)} = e^{iK_{cm}(r_2+r_1)}$

So the overall symmetry = symmetric! actually depends on H_{rel} .

and its eigenstate is given by

$$\Phi(r_1, r_2) \cdot \chi(r_1, r_2)$$

- ① Assume $s=0$, \Rightarrow boson $\Rightarrow \Phi(r_1, r_2)$ symmetric now if $\Psi_{nem}(r, \theta, \phi)$ and the exchange is equivalent to $r \rightarrow -r$ which is the same as $\Psi_{nem}(r \rightarrow -r, \theta, \phi) = (-1)^e \Psi_{nem}(r, \theta, \phi)$
 \Rightarrow to stay symmetric Ψ_{nem} only can have $e = \text{even}$!

Now let's say we have 2-electrons
 $S = 1/2$

As we discussed above we will have

$$\chi(s_1, s_2) = \begin{cases} \chi_s & \text{triplet } \uparrow\uparrow \\ \chi_a & \text{singlet } \downarrow\uparrow \end{cases}$$

~~Since~~ Since the total ψ must be antisymm.

we can say that $\psi_{\text{new}}(r, \theta, \phi) = \begin{cases} \text{antisymm} \\ \text{for } \chi_s \end{cases}$

$$\begin{cases} s=0 \mid l = \text{even} & \text{for } s=0 \\ l = \text{odd} & \text{for } s=1 \end{cases}$$

$$\begin{cases} \text{symm. for} \\ \chi_a \end{cases}$$

$$\text{overall } \boxed{l + s = \text{even}}$$

Now if ϕ_{α_1} and ϕ_{α_2} are the spatial wave functions

$$\uparrow\downarrow: \phi_s(12) = \frac{1}{\sqrt{2}} [\phi_{\alpha_1}(1) \phi_{\alpha_2}(2) + \phi_{\alpha_2}(1) \phi_{\alpha_1}(2)]$$

$$\uparrow\uparrow: \phi_a(12) = \frac{1}{\sqrt{2}} [\phi_{\alpha_1}(1) \phi_{\alpha_2}(2) - \phi_{\alpha_2}(1) \phi_{\alpha_1}(2)]$$

Assume now the electrons come very close to each other:

$$\left. \begin{aligned} \phi_{\alpha_1}(1) &\approx \phi_{\alpha_1}(2) \\ \phi_{\alpha_2}(1) &\approx \phi_{\alpha_2}(2) \end{aligned} \right\} \Rightarrow$$

$$\phi_a(12) \rightarrow 0$$

This means that probability of 2 electrons (triplet ones) $\uparrow \uparrow$ to come close is very small. Or it may look like they repel each other. This ~~is~~ effect is not b/c of their charge but rather from the symmetry consideration of having the overall antisymmetric wave function for fermions.

What about bosons?

in this case $\phi_S \approx \sqrt{2} \phi_{\alpha_1}(1) \phi_{\alpha_2}(2)$
 which is $\times 2$ over the average value.

Hence 2 non-interacting bosons love to
 (weakly)
 come together at the same space point if
 their eigenstate is symmetric.

So if $(\uparrow \downarrow)$ and $\hat{S} = 0$ they act like
 χ_a
 they attract each other.

This is the idea behind exchange "force".
 It looks like the fact of repulsion or
 attraction depends on what \hat{S}_{total} spin state our
 many-body system is.

This kind of interaction is known as
 exchange interaction and VERY important
 in condensed matter and especially
 in strongly correlated electronic matter.

This is purely quantum phenomena and
 is due to the fact we cannot label
 the particles in QM.

Read 18.7 on EXCITED STATE OF HE ATOM.