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The quantum Hall effect

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I. SCOPE

These problems apply undergraduate quantum mechanics to explain the quantum Hall effect. Their solution requires familiarity with the Schrödinger equation and its solutions for a one-dimensional harmonic oscillator, with the use of operator commutators to establish simultaneous eigenvalues, with periodic boundary conditions, and with the use of the magnetic vector potential in a Hamiltonian to describe the interaction of electrons with a magnetic field.

II. INTRODUCTION

The quantum Hall effect is one of the most striking and surprising developments to occur in physics in the last 20 years or so. Discovered[1] by von Klitzing, Dorda, and Pepper in 1980, the *integral* quantum Hall effect manifests itself as a series of *plateaus* in the Hall resistance, R_H, of materials containing *two-dimensional electron systems.* Spectacularly, these plateaus, shown in Fig. 1, occur when R_H is precisely given by

$$R_H=h/je^2=(25\,812.81/j)\Omega, \qquad (1)$$

where h is Planck's constant, e is the elementary charge, and j is any positive integer, $j=1,2,3,\ldots$. That the Hall resistance, employed for years as an indicator of the sign and density of free charge carriers in conductors and semiconductors, would exhibit behavior depending only on the fundamental constants of nature struck many as truly remarkable. For this discovery von Klitzing was awarded the 1985 Nobel Prize in Physics.[2] To date the simple result in Eq. (1) has been shown to be correct to within a few parts in 10^9! This has led to the adoption of the quantized Hall resistance as the international standard of resistance.[3]

Before presenting the problems themselves, a few introductory comments are required beginning with a definition of the Hall resistance. Imagine a conductor (not necessarily a simple metal) through which a current I is flowing. For simplicity take the shape of the conductor to be a long thin slab as in Fig. 2. If the conductor is not a superconductor, there is a voltage drop V along the the direction of current flow. The ratio of V to I is just the ordinary resistance $R=V/I$. Consider now trying to measure a voltage *transverse* to the current flow. By symmetry it should be clear that no such voltage will be observed; there is no reason for the charges to bunch up along one side of the bar. On the other hand, if a magnetic field is applied perpendicular to the current flow (and parallel to the thin dimension of the slab), then the Lorentz force will create just such an accumulation. This in turn will produce a voltage V_H across the slab, transverse to the direction of current flow. By definition the Hall resistance is this voltage divided by the current, I, $R_H=V_H/I$.

The notion of *two-dimensional* electrons also deserves comment. Restriction of electrons to two dimensions is an approximation to reality, just as in a game of billiards a two-dimensional model suffices as long as the third dimension is unemployed—billiard balls usually don't leave the table! In our case a two-dimensional system of electrons is simply one in which all motion occurs in a plane, the electrons being confined to the plane by some atomic forces that need not concern us. In fact, in most semiconductor-based 2D-electron systems the actual thickness of the 2D sheet is around 100 Å, suggesting that "two dimensional" really means "three dimensional, but very thin." The most common realization of such a system is known as a MOSFET (metal–oxide semiconductor field effect transistor). In such a device the electrons are confined to the plane interface between the semiconductor (usually silicon) and an oxide grown on top of it. The original discovery of the quantum Hall effect was made with a MOSFET device. More recently, a different system has been employed, the semiconductor heterojunction, in which the 2D electrons are confined at the interface between two different crystalline semiconductors. Nevertheless, the only important thing is that the electrons can move freely in a 2D plane but not in the perpendicular direction. Perhaps the most amazing aspect of the quantum Hall effect is its sublime indifference to the nature of the host material in which 2D electrons exist.

There are two fundamental ingredients to our understanding of the quantum Hall effect. The first is the so-

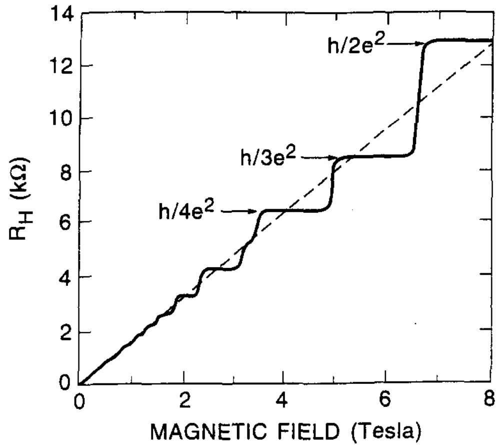

Fig. 1. Hall resistance of a typical two-dimensional electron gas versus magnetic field at a temperature $T=50$ mK. The dotted line is the classical result. (This figure is courtesy of M. A. Paalanen.)

called Landau quantization of states induced by the magnetic field on free 2D electron motion. This magnetic quantization produces the crucial *gaps* in the energy spectrum and is thoroughly examined in the problems that follow. The second ingredient is known as *localization.* While Landau quantization is within the scope of the typical undergraduate physics curriculum, localization physics is not. Consequently, only one of the two necessary keys to the quantum Hall effect will be examined in these exercises. The interested student is referred to the article by Halperin[4] for a description of localization physics.

III. THE PROBLEMS

Problem 1: The Hall resistance. Derive the classical formula for the Hall resistance where, as shown in Fig. 2 a slab-shaped conducting sample carries a current I parallel to its long axis. The slab's thickness and width are t and w, respectively, and a magnetic field B is applied perpendicular to the sample. The conductor contains a volume density N of free carriers with charge $-e$. Show that transverse to the current there exists a voltage of magnitude $V_H=IB/Nte$ giving a Hall resistance $R_H=B/Nte$.

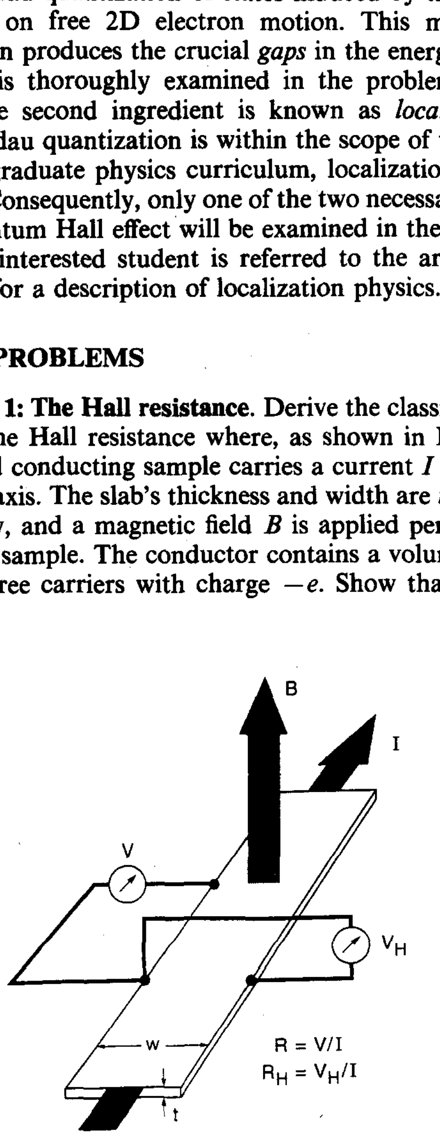

Fig. 2. Schematic diagram of a configuration for measuring voltage drop V along a conductor and the Hall voltage V_H transverse to the conduction current I in the presence of a magnetic field B. The resistance R and the Hall resistance R_H are determined from I, V, and V_H as indicated.

Taking the view that a 2D electron system is really just a thin 3D one, we can define the 2D areal carrier concentration as $N_s=Nt$ and obtain the 2D classical Hall formula:

$$R_H=B/N_s e. \tag{2}$$

As this formula suggests, a measurement of the Hall resistance determines both the sign and the concentration of the charge carriers in a conductor. This fact and the simplicity of the measurement itself make the Hall resistance one of the most frequently measured quantities in solid state physics. As Eq. (2) shows, the classical Hall resistance has a simple linear dependence on magnetic field.

Problem 2: 2D electrons in a magnetic field. We now turn to the quantum mechanics of 2D electrons in a magnetic field. Let the electron be confined to the x–y plane and the magnetic field $\mathbf{B}$ be parallel to the z axis. The Hamiltonian for such a free "2D" electron is given by

$$H=\frac{(\mathbf{p}+e\mathbf{A})^2}{2m}=\frac{(p_x+eA_x)^2+(p_y+eA_y)^2}{2m},$$

where $\mathbf{A}$ is the vector potential associated with the magnetic field $\mathbf{B}$, i.e., $\mathbf{B}=\nabla\times\mathbf{A}$.

(a) Given that $\mathbf{B}=B\hat{\mathbf{k}}$, show that both $\mathbf{A}=xB\hat{\mathbf{j}}$ and $\mathbf{A}=-yB\hat{\mathbf{i}}$ are equally good vector potentials ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are the usual Cartesian unit vectors).

(b) Using the "gauge" choice $\mathbf{A}=-yB\hat{\mathbf{i}}$, write down the Hamiltonian and show that the commutator $[H,p_x]=0$, implying that p_x is a good quantum number. For convenience define $k=p_x/\hbar$ and then argue that the wave functions $\Psi(x,y)$ are of the form:

$$\Psi(x,y)=e^{ikx}\phi(y).$$

(c) Show that the Schrödinger equation $H\Psi=E\Psi$ reduces to the *one-dimensional* problem:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{m\omega_c^2}{2}(y-y_0)^2\right)\phi(y)=E\phi(y),$$

with the definitions $\omega_c=eB/m$ and $y_0=\hbar k/eB$.

(d) Show by analogy to the one-dimensional simple harmonic oscillator that the eigenvalues E_n of H are independent of k and are given by:

$$E_n=(n+\tfrac{1}{2})\hbar\omega_c \text{ with } n=0,1,2,3,\ldots.$$

Thus the magnetic field has induced a great condensation of the continuous energy spectrum of a free particle in 2D into a discrete set of highly degenerate levels. These levels, known as *Landau levels,* are equally spaced by the *cyclotron energy,* $\hbar\omega_c$, which is itself proportional to the magnetic field strength. The *gaps* between the levels are void of electronic states. (It is interesting to note that for 3D electrons no such gaps occur, a fact the interested student can prove.)

(e) Show that the family of states belonging to the ground Landau level ($n=0$) are extended in the x direction but are gaussian, and therefore localized, in the y direction. For any k show that the gaussian is centered at $y=y_0$ and has approximate width $l_0=\sqrt{\hbar/eB}$. The parameter l_0 is called the *magnetic length.*

These eigenstates seem hardly related to the classical circular cyclotron motion one might have expected. If anything, our eigenstates appear to suggest a *linear* motion of the electron along the x axis! This confusion is due to the choice of gauge we made at the beginning. Were the symmetric gauge $\mathbf{A}=(\mathbf{r}\times\mathbf{B})/2$ chosen, eigenstates of manifest circular symmetry would have resulted.

Problem 3. Degeneracy of Landau states. There remains a critical aspect of Landau states that we must investigate—their *degeneracy.* For any given Landau index n there are many available k's. Just how many? Consider the rectangular region $0<x<L_x$, $0<y<L_y$ in which we wish to count states. The next question is what kind of boundary conditions to impose. While we could insist that the wave function actually vanish at the region's boundaries, it is equivalent for the purposes of counting states to use the so-called *periodic* boundary conditions in the x direction.

(a) By insisting that $\Psi(x=0,y)=\Psi(L_x,y)$ argue that the allowed k-values are $k_m=2\pi m/L_x$ with $m=0,1,2,\ldots$.

(b) Given the association of k and y_0 show that there is a maximum value M for the index m. Show that $M=L_xL_y/2\pi l_0^2$ giving the number of states *per unit area* in any Landau level as simply $N_0=eB/h$. The independence of this degeneracy N_0 from any material parameters (like the electron mass) is central to the phenomenon of the quantum Hall effect.

Problem 4. Quantum Hall effect. At this point we have all the ingredients for exposing one of the two major reasons for the quantization of the Hall resistance in 2D electron systems. Suppose many electrons are moving about in the 2D plane. If the temperature is low enough these electrons will reside in the lowest possible energy levels. But electrons must also obey the Pauli exclusion principle, which states that no two electrons can occupy the same quantum state. Therefore, at most $N_0=eB/h$ electrons per unit area can be fitted into the lowest Landau level with energy $E_0=\hbar\omega_c/2$. (This argument ignores the spin of the electron.[5]) Since the degeneracy N_0 depends linearly on the magnetic field, it should be clear that the energy at which the last electron is placed will depend upon the field. Obviously if B is high enough, *all* the electrons can be in the lowest Landau level.

Suppose that there are N_s electrons per unit area in a given sample, and that this is fixed. Adjust the magnetic field so that precisely j Landau levels are completely filled (at zero temperature). Show that the Hall resistance (cf. Problem 1) in this situation is given by $R_H=h/je^2$. These special values of R_H are precisely those at which the quantum Hall plateaus are observed!

IV. DISCUSSION

The quantum Hall effect is thus intimately connected with the discrete nature of the Landau spectrum. Exact filling of an integer number of Landau levels produces the correct Hall plateau values. Furthermore, a simple argument suggests why the ordinary, not Hall, resistance drops to vanishingly small values (an experimental fact not previously mentioned) when the Hall resistance is at one of the quantized values. Since the lowest j levels are completely filled, the only way for energy to be dissipated, the essence of resistivity, is by exciting electrons to the next Landau level. But this next level is separated by a gap $\hbar\omega_c$ from the highest occupied level. At low enough temperature there is simply not enough thermal energy around to bridge this gap.

While the existence of a discrete spectrum of Landau levels is clearly central to the quantum Hall effect, it is equally clear that it is not the whole story. Our simple "derivation" of the Hall plateau values ignores the fact that they are *plateaus*! The exact filling of the Landau states works only for specific values of the magnetic field, not over *ranges* of field.

The solution to this conundrum lies in the unavoidable imperfections in a real 2D electron system. Remarkably, it is simply the existence of disorder, not its detailed nature, that provides the key. One would think that the infinite variety of possible defects would destroy the *universality* of the quantum Hall effect. That it does not, helps to explain why such a striking effect was never predicted and, when discovered, created such amazement.

The role of disorder on 2D electron motion is beyond the scope of these problems. It is possible, however, to outline its role in the quantum Hall effect. Basically, the imperfections in the 2D plane, a nearby charged impurity atom, for example, can effectively create new electron states not at the exact Landau energies E_n. These states are often localized in the sense that an electron occupying one is trapped, orbiting the ion, for example, and can therefore not contribute to the transport of current through the sample. The current is carried only by those untrapped electrons at the exact Landau energies. It is now understood that as long as the last filled electronic states are localized ones, i.e., in the gaps of the ideal system, the Hall resistance maintains its quantized value, the index j being the number of ideal Landau levels that would be occupied at the given magnetic field. This is, of course, puzzling since it would seem that the trapped electrons should be subtracted from the density N_s and the Hall resistance would therefore deviate from the quantized value. In fact, the reduced number of conducting electrons miraculously "speed up" exactly enough to compensate for this and the quantized plateaus remain. A subtle argument based on gauge invariance has been proposed by Laughlin[6] to explain this result. The reader is again referred to Halperin's article[4] for further discussion.

Soon after the discovery of the integer quantum Hall effect came the observation[7] of *fractional* Hall plateaus. These new plateaus, seen only with the best samples, and at generally higher magnetic field and lower temperatures, correspond to the index j taking on fractional values, e.g., $j=1/3$. In many ways this fractional quantum Hall effect[8] is more stunning than the integer effect. The understanding of the integer effect is fundamentally based upon the existence of energy gaps in the electronic spectrum. These gaps have a simple explanation based upon the quantum mechanics of single free electrons in a magnetic field. Observation of additional plateaus outside the integer ones leads to the inescapable conclusion that additional gaps must exist. These additional gaps must derive from some physics outside single electron quantum mechanics.

It is now firmly established that the fractional Hall plateaus result from the mutual Coulomb interactions between electrons. These interactions create new many-body gaps in the energy spectrum. In fact, the fractional quantum Hall plateaus are signatures of new collective states of the many-electron system. These collective states are called *quantum liquids* and they possess novel correlations not previously anticipated. One is tempted to argue that while the integer quantum Hall effect perhaps *should* have been theoretically predicted, the far greater complexity of the many-electron problem makes it understandable that the fractional effect would only be discovered by surprise.

V. SOLUTIONS

Problem 1: The Hall resistance. Refer to Fig. 2 for geometric details. Assume that the current flow down the bar can be thought of as due to a uniform drift of the charge carriers with some velocity v_d. The current density j, i.e., the current per unit cross-sectional area, is just $j=I/tw=Nev_d$. The effect of the magnetic field is to push the charges sideways with a Lorentz force of magnitude $F_L=ev_dB$. Since the charges can not escape out the sides of the bar they accumulate at one side and are depleted on the opposite side. This charge separation produces an electric field E opposing further accumulation. In equilibrium this electric field precisely balances the Lorentz force so that $E=F_L/e=v_dB$. In turn, the electric field produces a *Hall* voltage V_H across the sample given by $V_H=Ew=wv_dB$. Using the relation between the current density j and the drift velocity, we obtain $V_H=(B/Nte)I$. Defining the Hall resistance $R_H=V_H/I$ we get the desired result $R_H=B/Nte$.

For two-dimensional systems an areal density N_s of charge is often more useful than a volume density N. Given the layer thickness t, it follows that $N_s=Nt$ and, in these terms, that $R_H=B/N_se$.

Problem 2: 2D electrons in a magnetic field. (a) The curl operation can be evaluated using the determinant method:

$$\nabla\times\mathbf{A}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}.$$

Direct substitution of either $\mathbf{A}=xB\hat{\mathbf{j}}$ or $\mathbf{A}=-yB\hat{\mathbf{i}}$ gives $\mathbf{B}=B\hat{\mathbf{k}}$.

(b) Substituting $\mathbf{A}=-yB\hat{\mathbf{i}}$ into the Hamiltonian $H=(\mathbf{p}+\mathbf{A})^2/2m$ yields (in two dimensions),

$$H=p_y^2/2m+(p_x-eyB)^2/2m.$$

Since the coordinate x does not appear in H, and since we know the commutators $[y,p_x]=[p_y,p_x]=0$, it follows that $[H,p_x]=0$. This means that p_x is a good quantum number and may be regarded as a constant parameter. We define $k=p_x/\hbar$. The eigenfunctions for the problem must now simultaneously satisfy:

$$p_x\Psi(x,y)=\hbar k\Psi(x,y),$$

$$H\Psi(x,y)=E\Psi(x,y).$$

Since the operator p_x is given by $-i\hbar\partial/\partial x$, the first equation yields the desired result:

$$\Psi(x,y)=e^{ikx}\phi(y).$$

(c) The second eigenvalue equation is the Schrödinger equation, and its solutions determine the energy spectrum. We have

$$\begin{aligned}H\Psi&=[(p_x-eyB)^2+p_y^2]e^{ikx}\phi(y)/2m,\\ &=e^{ikx}[(\hbar k-eyB)^2+p_y^2]\phi(y)/2m,\\ &=e^{ikx}\left(\frac{e^2B^2}{2m}(y-\hbar k/eB)^2+\frac{p_y^2}{2m}\right)\phi(y),\\ &=e^{ikx}\left(\frac{m\omega_c^2}{2}(y-y_0)^2+\frac{p_y^2}{2m}\right)\phi(y),\end{aligned}$$

where $\omega_c=eB/m$ and $y_0=\hbar k/eB$. The right-hand side must equal $E\Psi=Ee^{ikx}\phi(y)$. Thus the required differential equation is obtained (after replacing p_y by $-i\hbar\partial/\partial y$):

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}+\frac{m\omega_c^2}{2}(y-y_0)^2\right)\phi(y)=E\phi(y).$$

(d) Begin by recalling the ordinary one-dimensional Schrödinger equation for a simple harmonic oscillator with frequency ω:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{m\omega^2}{2}z^2\right)\phi(z)=E\phi(z),$$

where z is a dummy coordinate. Our problem looks very similar except we have a term $(y-y_0)^2$ instead of just y^2. But this simply means that the center of the oscillations is at $y=y_0$ rather than at $y=0$ as you can see by making the change of variable $z=y-y_0$. For any given y_0, i.e., for any k, our problem is *exactly* like the corresponding 1D simple harmonic oscillator. Most important, the energy levels for the two problems must be the same. From your knowledge of simple harmonic oscillators it should be no surprise then that

$$E_n=(n+\tfrac{1}{2})\hbar\omega_c \text{ with } n=0,1,2,\ldots .$$

This is a remarkable result: The allowed energy levels for a *free* 2D electron moving in a magnetic field are identical to a fictitious 1D simple harmonic oscillator. Note these levels are *discrete*; without the magnetic field the electron's energy is a continuous variable.

Note also that the energy levels do not depend on the value of y_0 or, therefore, k. The "momentum" k in the x direction creates no kinetic energy! Each value of n can have, apparently, any value of k and thus the energy levels are highly degenerate.

(e) Since you have already solved Problem 2(b), you know that the x dependence of all Landau level states is of the form of e^{ikx}. Since the probability of finding the electron at any position is determined by $|\Psi|^2$, there is no variation in the x direction. Hence, all values of x are equally likely. We say the state is *extended* in this direction.

For the y direction it suffices to solve the 1D Schrödinger equation from Problem 2(d). Since you are looking for a gaussian solution, the easiest approach is to substitute in a trial wave function $\phi(z) = Ce^{-az^2}$, where C and a are undetermined constants. The constant C can be ignored; it is only important for normalizing the wave function. By direct substitution you find:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}+\frac{m\omega^2}{2}z^2\right)\phi(z)$$

$$=e^{-az^2}\left(-\frac{\hbar^2}{2m}(4z^2a^2-2a)+\frac{m\omega^2}{2}z^2\right).$$

Since this must equal $E\phi(z) = Ee^{-az^2}$, the terms proportional to z^2 must vanish. This will determine the constant a and you will find $a=m\omega/2\hbar$. Note also that with this value of a one gets $E=\hbar\omega/2$, the correct energy for the lowest Landau level. Now that you know the lowest state is a Gaussian in the z direction you can estimate how close to $z=0$ the electron remains. It should be clear that if z is much greater than $a^{-1/2}$, the probability becomes very small. Substituting in the definition of ω_c gives $a^{-1/2} = \sqrt{2}l_0$ with $l_0 = \sqrt{\hbar/eB}$. Interestingly, the magnetic length l_0 depends only on B not the mass m. At $B=1$ T this length is $l_0=256.6$ Å.

Recall that the dummy variable z was used to simplify things. Going back to the y coordinate shows that each gaussian is centered at $y=y_0$; y_0 itself depends upon k.

Problem 3. Degeneracy of Landau states. (a) Since $\Psi(x,y)=e^{ikx}\phi(y)$ the periodicity requirement becomes $e^{ikL_x} = 1$ from which we conclude kL_x must be a multiple of 2π. Thus $k_m=2\pi m/L_x$ with $m=0,1,2,\ldots$.

(b) The value of y_0 is determined by k from the relation $y_0=\hbar k/eB$. The electron wave function is centered at $y=y_0$. Clearly, the value of y_0 can not be greater than L_y, the extent of the sample in the y direction. From the requirement that $y_0 < L_y$ we get $k < eBL_y/\hbar$. It therefore follows that there is a maximum value M of the index m given by $M = L_x k_{max}/2\pi = L_x L_y/2\pi l_0^2$ where l_0 is the magnetic length $l_0 = \sqrt{\hbar/eB}$. The number M is the total number of available states in each Landau level in the available sample area. This determines the degeneracy per unit area to be $N_0=eB/h$.

Problem 4. Quantum Hall effect. Since each Landau level can hold the same total number of electrons, if there are exactly j levels completely filled it must be true that $N_s=jN_0$. But the Hall resistance is $R_H=B/N_s e$. Thus for this special circumstance $R_H=B/(ejN_0)$, or using the definition of N_0, $R_H=h/je^2$.

[1] K. von Klitzing, G. Dorda, and M. Pepper, "New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance," Phys. Rev. Lett. **45**, 494–497 (1980).

[2] K. von Klitzing, "The quantized Hall effect," Rev. Mod. Phys. **58**, 519–531 (1986). This is von Klitzing's Nobel lecture.

[3] B. N. Taylor, "New measurement standards," Physics Today **42**, 23–26 (1989).

[4] B. I. Halperin, "The quantized Hall effect," Sci. Am. **254**, 52–60 (1986).

[5] If there were no spin, the correct degeneracy of each Landau level would be eB/h as Problem 3 asserts. Hall plateaus would be observed at h/je^2 for $j=1,2,3\ldots$. With the addition of spin each Landau level becomes a doublet. Each component of the doublet has degeneracy eB/h making the entire Landau level degeneracy twice that, or $2eB/h$. If the Zeeman splitting of the two spin levels is not resolved, say because the temperature is too high, then only *even* Hall plateaus will be found: h/je^2 for $j=2,4,6,\ldots$. If the spin is resolved then the plateaus at $j=1,3,5\ldots$ will also appear. Usually one is part way between these extremes. Since the Zeeman splitting is linear in B generally one can resolve the spin at high B but not at low B. This is the case in Fig. 1. While $h/3e^2$ is clearly seen, $h/5e^2$ is only a shoulder appearing around 3.5 T. Below 3 T all the plateaus are "even."

[6] R. B. Laughlin, "Quantized Hall conductivity in two dimensions," Phys. Rev. **B23**, 5632–5633 (1981).

[7] D. C. Tsui, H. L. Stormer and A. C. Gossard, "Two-dimensional magnetotransport in the extreme quantum limit," Phys. Rev. Lett. **48**, 1559–1562 (1982).

[8] J. P. Eisenstein and H. L. Stormer, "The fractional quantum Hall effect," Science **248**, 1510–1516 (1990).

THE FOUCAULT PENDULUM

That was when I saw the Pendulum.

The sphere, hanging from a long wire set into the ceiling of the choir, swayed back and forth with isochronal majesty.

I knew—but anyone could have sensed it in the magic of that serene breathing—that the period was governed by the square root of the length of the wire and by π, that number which, however irrational to sublunar minds, through a higher rationality binds the circumference and diameter of all possible circles. The time it took the sphere to swing from end to end was determined by an arcane conspiracy between the most timeless of measures: the singularity of the point of suspension, the duality of the plane's dimensions, the triadic beginning of π, the secret quadratic nature of the root, and the unnumbered perfection of the circle itself.

Umberto Eco, *Foucault's Pendulum* (Harcourt Brace Jovanovich, New York, 1989), p. 3.