Lecture 1

Time independent perturbation theory.

The essence of the method: Split Hamiltonian into 2 parts: $H_0 + H_1$, where $H_0$ is large and exactly solvable = unperturbed part and $H_1$ = perturbation. Changes in eigenstates and eigenvalues is treated as a power series in the perturbation parameter.

Let's develop this calculation scheme for non-degenerate and degenerate case separately.

A. Non-degenerate case: or 1 eigenstate have one eigenvalue.

\[ H = H^{(0)} + H^{(1)} \] we solve \[ H \phi_n = E_n \phi_n \]
we assume we do know $\phi_n^{(0)}$ and $E_n^{(0)}$

Our goal is to express the complete solution as:

\[ E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots \]

\[ \phi_n = \phi_n^{(0)} + \lambda \phi_n^{(1)} + \lambda^2 \phi_n^{(2)} + \ldots \]

For simplicity, let's work out 1st for $(E_n^{(1)}, \phi_n^{(1)})$ and $(E_n^{(2)}, \phi_n^{(2)})$ or

\[ \begin{bmatrix} H^{(0)} + \lambda H^{(1)} \end{bmatrix} \begin{bmatrix} \phi_n^{(0)} + \lambda \phi_n^{(1)} + \lambda^2 \phi_n^{(2)} + \ldots \end{bmatrix} = \begin{bmatrix} E_n^{(0)} \phi_n^{(0)} + \lambda E_n^{(1)} \phi_n^{(1)} + \lambda^2 E_n^{(2)} \phi_n^{(2)} + \ldots \end{bmatrix} \]

All the terms with equal power in $\lambda$ should be set to 0.
We assume we know the exact solution of the unperturbed case i.e. 
\[ H^{(0)} \phi_n^{(0)} = E_n^{(0)} \phi_n^{(0)} \]

Let's solve for \( \lambda^{(1)} \) for this purpose we select the basis vectors \( \{ \phi_m^{(0)} \} = \phi_m^{(0)} \) and as such 
\[ \phi_n^{(1)} = \sum_m C_m^{(1)} \phi_m^{(0)} \]

We can do this since the basis vectors \( \phi_m^{(0)} \) are known and they are good (orthogonal etc.) 

\[ \sum_m C_m^{(1)} H^{(0)} \phi_m^{(0)} + H^{(1)} \phi_n^{(0)} = E_n^{(0)} \sum_m C_m^{(1)} \phi_m^{(0)} + E_n^{(1)} \phi_n^{(0)} \]

\[ \Rightarrow \]

\[ \phi_n^{(0)*} \int dV \sum_m C_m^{(1)} \left( E_m^{(0)} - E_n^{(0)} \right) \phi_m^{(0)} = \left( E_n^{(1)} - H^{(1)} \phi_n^{(0)} \right) \phi_n^{(0)} \]

\[ \Rightarrow \]

\[ \int dV \sum_m C_m^{(1)} \left( E_m^{(0)} - E_n^{(0)} \right) \phi_r^{(0)*} \phi_m^{(0)} = \xi E_n^{(1)} \phi_n^{(0)*} \phi_n^{(0)} \]

\[ \langle r | (E_r^{(0)} - E_n^{(0)}) \rangle = E_n^{(1)} \delta_{rn} - H_r^{(1)} \]

Here 
\[ \mu_r^{(1)} = \int_{-\infty}^{\infty} \phi_r^{(0)*} H^{(1)} \phi_r^{(0)} dV \]

\[ \int_{-\infty}^{\infty} \phi_r^{(0)*} \phi_m^{(0)} dV = \delta_{rm} \]

To determine \( E_n^{(1)} \) set \( r = n \)

\[ \langle n | (E_n^{(0)} - E_n^{(0)}) \rangle = E_n^{(1)} - \langle \phi_n^{(0)} | H^{(1)} | \phi_n^{(0)} \rangle \]

\[ E_n^{(1)} = \langle \phi_n^{(0)} | H^{(1)} | \phi_n^{(0)} \rangle \approx H_n^{(1)} \]

And for eigenstates: 
\[ \langle n | \lambda^{(1)} | n \rangle = \frac{E_n^{(1)} \delta_{nn} - H_n^{(1)}}{E_n^{(0)} - E_n^{(0)}} = \frac{H_n^{(1)}, \delta_{nn} - H_n^{(1)}}{E_n^{(0)} - E_n^{(0)}} \]

\[ = \frac{H_n^{(1)} - H_n^{(0)}}{E_r^{(0)} - E_n^{(0)}} = \frac{E_n^{(0)} - E_r^{(0)}}{E_n^{(0)} - E_n^{(0)}} \]

\[ = \frac{\langle \phi_n^{(0)} | H^{(1)} | \phi_n^{(0)} \rangle}{E_n^{(0)} - E_r^{(0)}} \]
Notice that we can write down

\[ \Phi_n^{(1)} = \Phi_n^{(0)} + \Phi_n^{(1)} + \sum_{m \neq n} \frac{H_{mn}}{E_n^{(0)} - E_m^{(0)}} \Phi_m^{(0)} \]

So all our equations for \( E_n^{(1)} \) and \( c_n^{(1)} \) remain the same.

Recall that:

\[ \Phi_n^{(1)} = \Phi_n^{(0)} + \Phi_n^{(1)} + \sum_{m \neq n} \frac{H_{mn}}{E_n^{(0)} - E_m^{(0)}} \Phi_m^{(0)} \]

\[ E_n^{(1)} = E_n^{(0)} + \sum_{m \neq n} \frac{H_{mn}}{E_n^{(0)} - E_m^{(0)}} \]

\[ \langle \Phi_n^{(1)} | \Phi_n^{(1)} \rangle \approx \langle \Phi_n^{(0)} | \Phi_n^{(1)} \rangle + \sum_{m \neq n} \frac{\langle \Phi_m^{(0)} | H_{mn} | \Phi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \]

In other words, the change in the energy caused by \( H^{(1)} \) should be much smaller than the distance between levels \( |E_n^{(0)} - E_m^{(0)}| \).

So the perturbation doesn't have to be small itself, it needs to be much smaller than the spacing between the states of the unperturbed (ground state) Hamiltonian.

For the second order we get:

\[ E_n^{(2)} = \sum_{m \neq n} \sum_{m \neq n} \sum_{m \neq n} \frac{\langle \Phi_m^{(0)} | H_{mn} | \Phi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \]

This will be a part of your 1st homework.

Notice:

1. The ground state is lowered in the 2nd order since \( \frac{1}{E_n^{(0)} - E_m^{(0)}} \) is \( > 0 \).
2. Energies are pushed apart in the 2nd order.
3. Because of \( \frac{1}{E_n^{(0)} - E_m^{(0)}} \), if \( E_n^{(0)} \approx E_m^{(0)} \), the change is huge.
One of the problems with \( n = 2 \) and up is that we need to do summation over the infinite series. Enter the Dalgarno and Lewis method.

Read pp. 336 - 337. Here I just quote the final result:

\[
E_n^{(2)} = \langle \Psi \mid H^{(1)} \mid \Phi_n \rangle - E_n^{(1)} \langle \Psi \mid \Phi_n \rangle
\]

\[
E_n^{(3)} = \langle \Psi \mid H^{(1)} \mid \Phi_n \rangle E_n^{(1)} + 2 E_n^{(2)} \langle \Psi \mid \Phi_n \rangle - E_n^{(1)} \langle \Psi \mid \Phi_n \rangle
\]

By looking at \( E_n^{(2)} \) we see we need only calculate 2 integrals instead of \( 3n \), i.e. \( \langle \Psi \mid H^{(1)} \mid \Phi_n \rangle \) and \( \langle \Psi \mid \Phi_n \rangle \).

Q: The question is what to do with \( \Phi_n \)?

A: This method works superbly well if \( H^{(1)} \) the perturbation has no differential term(s). In this case

\[
F_n(x) = \int_{-\infty}^{\infty} \frac{1}{f_n(x)} \left[ \frac{2m}{h^2} \int_{-\infty}^{\infty} \left( H^{(1)} - E_n^{(1)} \right) \Phi_n(x) \right] \phi_n(x) dx
\]

Let's work out the problem of a particle in the box \( H = H_0 + H^{(1)} \) where

\[
H_0 = \frac{p^2}{2m} + V(x)
\]

\[
H^{(1)} = \begin{cases} 0 & 1x1 \leq \frac{\pi}{2} \\ \infty & 1x1 > \frac{\pi}{2} \end{cases}
\]

and we will apply a perturbation

\[
H^{(1)} = \begin{cases} 0 & 1x1 \leq \frac{\pi}{2} \\ \infty & 1x1 > \frac{\pi}{2} \end{cases}
\]

The gradient field...
Recall from your QM the unperturbed states are:
\[ \phi_0^{(0)} = \sqrt{\frac{2}{\pi}} \cos x \quad \phi_1^{(0)} = \sqrt{\frac{2}{\pi}} \sin x \]
\[ \phi_2^{(0)} = \sqrt{\frac{2}{\pi}} \sin 2x \quad E_0^{(0)} = (n + 1) \frac{2 - \frac{h^2}{4m}}{2m} \]

Let's apply the D-L method.

\[ F_0(x) = \frac{2m}{h^2} \int x \cos 2x \left( \int_0^x \cos^2 \xi \, d\xi \right) \, dx = \]
\[ = \frac{\alpha m}{h^2} \left[ \int \frac{1}{2} \sec^2 x \int \frac{\xi^2}{2} + \frac{1}{2} \left( 2 \xi \sin 2\xi + \cos 2\xi \right) \right]_0^x \, dx \]
\[ = \frac{\alpha m}{2h^2} \left[ x^2 \sec^2 x + 2x \tan x + 1 - \frac{1}{2} \sec^2 x - \sec^2 x \left( a^2 + a \sin 2a + \frac{1}{2} \cos 2a \right) \right]_0^x \, dx \]
\[ \phi_0^{(0)}(a) = 0 \implies \sqrt{\frac{2}{\pi}} \cos(a) = 0 \implies a = \pm \frac{\pi}{2} \]
\[ F_0(a) = F_0(a = -\frac{\pi}{2}) = \frac{\alpha m}{2h^2} \left[ (x^2 - \frac{\pi^2}{4}) \tan x + x \right] \]

For \[ \langle 0 | F_0 | 1 \rangle = \frac{\langle 0 | H^{(1)} | 1 \rangle}{E_0^{(0)} - E_1^{(0)}} = \int_{x = -\frac{\pi}{2}}^{+\frac{\pi}{2}} \sqrt{\frac{2}{\pi}} \cos x \left( \frac{h^2}{2m} - \frac{4h^2}{2m} \right) \]

\[ \sqrt{\frac{2}{\pi}} \sin 2x \, dx = -\frac{2\alpha \lambda}{21\pi}; \text{ Finally} \]

\[ E_0^{(1)} = \frac{\langle 0 | H^{(1)} | 1 \rangle}{E_0^{(0)} - E_1^{(0)}} \implies E_0^{(1)} = \frac{2\alpha \lambda^2 \frac{m}{h^2}}{35\pi \frac{m}{h^2}} \]

Next \[ \langle 0 | F_0 | 2 \rangle = 0 \text{ and } \langle 0 | F_0 | 1 \rangle = 2 \alpha m \frac{\pi^2}{35\pi} \]

\[ E_0^{(2)} = -\frac{2\alpha \lambda^2 m}{35\pi \frac{m}{h^2}} - \frac{2\alpha \lambda^2 m}{35\pi \frac{m}{h^2}} = -\frac{2.1097 \alpha \lambda^2 m}{\frac{h^2}{2m}} \] (D-L Method)

\[ E_0^{(2)} = -\frac{2\alpha \lambda^2 m}{35\pi \frac{m}{h^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( x^2 - \frac{\pi^2}{4} \right) \tan x + x \, dx = -\left( \frac{\pi^2}{2} - \frac{\pi^2}{2} \right) \frac{\lambda^2 \frac{m}{h^2}}{21\pi \frac{m}{h^2}} \]
Before switching to the applications we need to recap the second quantisation:

(See Lecture 20* posted on-line)

1. Perturbed linear harmonic oscillator.

What will happen when we apply an external electric field to our crystal with ions bound to the harmonic potential?

\[ V(x) = \frac{m\omega^2 x^2}{2} - eEx \]

In other words,

\[ H = H_0 + H_1, \quad H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \] and \[ H_1 = -eEx \]

for \( n = 0 \) we get:

\[ E_n = (n + \frac{1}{2}) \hbar \omega \quad n = 0, 1, 2, \ldots \]

\[ \psi_n = N_n H_n(s) e^{-\frac{s^2}{2}} \]

\[ N_n = \left( \frac{\alpha}{\sqrt{2\pi} \, n!} \right)^{1/2} \quad \text{and} \quad H_n(s) \text{ are Hermite polynomials.} \]

This problem is nice b/c the exact solution for \( H = H(0) + \lambda H(1) \) is also well known.

1) Exact way:

\[ H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2 - \lambda eEx \]

\[ \text{new variable} \quad x = \frac{y}{\sqrt{2}} \Rightarrow \]

\[ H = \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 (y^2 + 2\alpha y + \alpha^2) - \lambda eEx \]

\[ = \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 (y^2 + \alpha^2) - \lambda eEx \]  
\[ + \frac{1}{2} m\omega^2 \alpha \]

\[ \frac{1}{2} m\omega^2 \alpha - \lambda eE \]

\[ = 0 \Rightarrow \alpha = \frac{\lambda eE}{m\omega^2} \]

Therefore, \[ H = \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2 - \lambda eEx \]

\[ \text{harmonic oscillator} \Rightarrow E_n(\text{exact}) = (n + \frac{1}{2}) \hbar \omega - \frac{\lambda^2 e^2 E^2}{2m \omega^2} \]  

\[ \text{constant shift inquiry} \]

\[ \text{harmonic oscillator} \Rightarrow E_n(\text{exact}) = (n + \frac{1}{2}) \hbar \omega - \frac{\lambda^2 e^2 E^2}{2m \omega^2} \]
the change in the energy is the 2nd order process.

2) Apply the perturbation theory:

\[ E_n^{(1)} = \langle n | H^{(1)} | n \rangle \equiv H_{nn}^{(1)} = \int \phi_n^{(0)*} H^{(1)} \phi_n^{(0)} dx \]

The solution is much easier if we move to \( a \) and \( a^+ \):

\[ x = \sqrt{\frac{\hbar}{2m \omega}} (a + a^+) \quad \text{and} \quad a |n\rangle = \sqrt{n} |n-1\rangle \quad a^+ |n\rangle = \sqrt{n+1} |n+1\rangle \]

and \( \langle n | n \rangle = \delta_{nn} \)

for \( E_n^{(1)} = \langle n | H^{(1)} | n \rangle = -\epsilon E \left( \frac{\hbar}{2m \omega} \right)^{1/2} \sqrt{n} \)

\[ \sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle = 0 \]

\[ E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n | H^{(1)} | m \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \]

\[ = \frac{\epsilon^2 E^2}{2m \omega^2} \sum_{m \neq n} \left( \frac{\sqrt{m} \delta_{m,n-1} + \sqrt{m+1} \delta_{m,n+1}}{\sqrt{n-\frac{1}{2}} - (n+\frac{1}{2})} \right) \]

\[ = \frac{\epsilon^2 E^2}{2m \omega^2} \]

This is mass!

\[ \text{and thus } E_n = E_n^{(0)} \text{ (exact)} + \frac{\epsilon^2 E^2}{2m \omega^2} \]

So the results coincide
Hyperfine splitting in hydrogen atom

Start this section by reading FLP VIII p. 12-1 Ch. 12.

The shortcut version will be presented in-class.

Splitting of energy due to the extra interaction of the electron orbital current \( \leftrightarrow \) a magnetic field with the nuclear magnetic dipole is called the fine structure.

Hyperfine splitting is very small splitting of the spectral lines of \( H \).

From electrodynamics the magnetic field of the magnetic dipole at the center of a sphere is:

\[
B(r) = \frac{\mu_0}{4\pi r^3} \left[ s(m \cdot r) r - m \right] + \frac{2}{3} \mu_m \delta^3(r)
\]

The energy of the dipole in the magnetic field \( \vec{B} \):

\[
H = -m \cdot \vec{B}
\]

Now we have two magnetic dipoles \( m_1 \) and \( m_2 \):

\[
H = -\frac{\mu_0}{4\pi r^3} \left[ \vec{m}_1 \cdot \vec{r} \cdot \vec{m}_2 - m_1 \cdot m_2 \right] - \frac{2}{3} \mu_0 m_1 \cdot m_2 \delta^3(r)
\]

From the perturbation theory:

\[
E_0^{(1)} = \langle \Psi_H | H^{(1)} | \Psi_H \rangle = \text{Hund} = \int \phi_0^* H \cdot \phi_0 d\vec{r}
\]

where \( \phi_0 = \left( \frac{1}{\pi a_0^3} \right)^{1/2} e^{-\frac{\vec{r}^2}{a_0}} \) where \( a_0 = a_{\text{Bohr}} \) radius

Since \( \phi_0 \) is spherically symmetric \( \equiv S \)-state!

we can conclude

\[
E_0^{(1)} = \int_{-\infty}^{0} m \cdot r \cdot \phi_0^* \cdot \phi_0 d\theta = 0
\]

\[
E_0^{(1)} = -\frac{2}{3} \mu_0 \left( \vec{m}_1 \cdot \vec{m}_2 \right) \phi_0^* \frac{2}{3} \frac{\delta^3(r)}{r} = -\frac{2}{3} \frac{\mu_0}{\pi a_0^3} \left( \vec{m}_1 \cdot \vec{m}_2 \right)
\]
\[ m_1 = \gamma_p s_p \quad m_2 = -\gamma_e s_e \quad \text{or} \quad m_1 = \gamma_e s_e \quad m_2 = -\gamma_p s_p \]

\[ E_1 = \frac{2}{3} \frac{\mu_0 e^2}{\hbar q_0^3} \gamma_p \gamma_e <s_e \cdot s_p> \]

the good quantum number is \( j = s_1 + s_2 \) or

\[ j^2 = s_1^2 + s_2^2 + 2 s_1 s_2 \implies j^2 - s_1^2 - s_2^2 = s_1 \cdot s_2 \]

At \( j = 1 \) or \( j = 0 \)

\[ \text{triplet} \quad \text{singlet} \]

\[ s_1^2 = s_1 (s_1 + 1) = \frac{1}{2} (j + 1) = \frac{3}{2} \hbar^2 \]

\[ s_2^2 = s_2 (s_2 + 1) = \frac{1}{2} (j + 1) = \frac{1}{2} \hbar^2 \]

or \( <s_e \cdot s_p> = \begin{cases} 1 - \frac{3}{4} & = \frac{1}{4} \hbar^2 \\ 0 - \frac{3}{4} & = -\frac{3}{4} \hbar^2 \end{cases} \]

Singlet is better since

The energy gap between the singlet and triplet:

\[ \Delta E = \frac{2 \mu_0 e^4}{3 \hbar q_0^3} \gamma_p \gamma_e \approx 5.88 \times 10^{-6} \text{ eV} \]

\[ \approx 5.8 \text{ meV} \]

and if we pump it from singlet to triplet we need to use a laser with

\[ h \omega_c = \frac{\Delta E}{\hbar} = 1422.8 \text{ MHz} \]

![Image](image.png)

Looks like the perturbation term works very well since the energy \( \Delta E \) is indeed very small w.r.t. electron energy at \( \approx 10 \text{ eV} \).
Theory for degenerate levels

If the levels are degenerate, i.e., $E^{(0)}_n = E^{(0)}_m$, all the perturbation terms become problematic
since $\frac{1}{E^{(0)}_n - E^{(0)}_m}$. So we need to develop a special case for the degenerate levels.

Recall degeneracy of the ground states stems from the symmetry of a Hamiltonian.

⇒ If the perturbation breaks the symmetry it will remove the degeneracy and our non-degenerate case is OK.

Let's start assuming that eigenvalues of the unperturbed Hamiltonian are $m$-fold degenerate, i.e., we have $m$ linearly independent $\phi^{(0)}_1, \phi^{(0)}_2, \ldots, \phi^{(0)}_m$. Corresponding to the same $E^{(0)}_1 = E^{(0)}_2 = \ldots = E^{(0)}_m = E^{(0)}$.

Since $H^{(0)}$ cannot select the "correct" ground state or eigenstate we form

$$\phi^{(0)} = \sum_{i=1}^{m} C_i \phi^{(0)}_i$$

and expect that $H^{(0)}$ will "select" the proper ground state.

Recall for the non-degenerate case:

$$\begin{align*}
\phi & = \phi^{(0)} + \lambda \phi^{(1)} + \lambda^2 \phi^{(2)} + \ldots \\
E & = E^{(0)} + \lambda E^{(1)} + \lambda^2 E^{(2)} + \ldots
\end{align*}$$

$$H^{(0)} H \phi = E \phi \Rightarrow \left[ H^{(0)} + \lambda H^{(1)} \right] \phi = E \phi$$

$$\begin{align*}
(H^{(0)} + \lambda H^{(1)}) (\phi^{(0)} + \lambda \phi^{(1)} + \ldots) &= (E^{(0)} + \lambda E^{(1)} + \ldots) \\
(\phi^{(0)} + \lambda \phi^{(1)} + \ldots) &= E \phi^{(0)} + \lambda E \phi^{(1)} + \ldots
\end{align*}$$
1. $H^{(0)} \phi^{(0)} = E^{(0)} \phi^{(0)}$

2. $H^{(0)} \phi^{(1)} + H^{(1)} \phi^{(0)} = E^{(0)} \phi^{(1)} + E^{(1)} \phi^{(0)}$

since we don't know anything about the perturbation (no constraint) we can make the solution in the following form:

$$\phi^{(1)} = \sum_i a_i \phi_i^{(0)}$$

$$\int_{\phi} \phi^* \sum_i \left( E_i^{(0)} - E^{(0)} \right) a_i \phi_i^{(0)} = \left( E^{(1)} - H^{(1)} \right) \sum_i c_i \phi_i^{(0)}$$

$$ar \left( E^{(0)} - E^{(0)} \right) = \sum_i \left( E^{(1)} \phi_i - \sum H_{ri} c_i \right)$$

for $r \leq m \ E_r^{(0)} = E^{(0)}$ (since the levels are degenerate)

$$0 = E^{(1)} \phi_r - \sum_{i=1}^{m} H_{ri} c_i \quad \text{or} \quad$$

$$\sum_{i=1}^{m} \left( H_{ri}^{(0)} - E^{(1)} \delta_{ri} \right) c_i = 0 \quad r = 0, 1, 2, \ldots$$

This means we have $m$ equations for $c_i$

As usual for a non-trivial case:

$$\begin{vmatrix} H_{11}^{(1)} - E^{(1)} & \cdots & H_{1m}^{(1)} \\ \vdots & \ddots & \vdots \\ H_{m1}^{(1)} & \cdots & H_{mm}^{(1)} - E^{(1)} \end{vmatrix} = 0$$

if all $E_1^{(1)}, \ldots, E_m^{(1)}$ are distinct, the perturbation removes degeneracy completely

Otherwise, we have the case of partial removal.
The eigenfunction can be found by setting $c_r = 0$ for $r > m$ or:

$$a_r = \frac{\sum_{i=1}^{m} H_{ri} c_i}{E^{(0)} - E_r^{(0)}} \quad \text{for } r > m$$

To make it clear, we use $r \leq m$ equations to compute $E^{(1)}$ and the remaining $r > m$ to compute the cost of the $\phi^{(1)}$ eigenvector.

Thus we have $r > m$ eqn. Therefore we can set $a_r = 0$ for $r > m$:

$$E_k^{(1)} = E_k^{(0)} + \lambda E_k^{(1)}$$

$$\phi_k^{(1)} = \phi_k^{(0)} + \lambda \sum_{r=m}^{m} \frac{\sum_{i=1}^{m} H_{ri} c_i}{(E^{(0)} - E_r^{(0)})} \phi_r^{(0)}$$

**Example:** 2D Harmonic oscillator:

$$H^{(0)} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{k}{2} (x^2 + y^2)$$

with eigenstates $|nx \hbar \text{x}, my \hbar \text{y}>$

Let's perturb the oscillator with a static potential:

$$H^{(1)} = \lambda xy$$

for the two-fold degenerate state $1017$ and $110>.$

To simplify the calculation, we introduce $a, a^+$ and $b, b^+$ for $x,$ and $y$ via this:

$$\begin{align*}
a &= \frac{1}{\sqrt{2}} \left( a x + \frac{1}{\alpha} \frac{d}{dx} \right) & a^+ &= \frac{1}{\sqrt{2}} \left( a x - \frac{1}{\alpha} \frac{d}{dx} \right) \\
b &= \frac{1}{\sqrt{2}} \left( a y + \frac{1}{\beta} \frac{d}{dy} \right) & b^+ &= \frac{1}{\sqrt{2}} \left( a y - \frac{1}{\beta} \frac{d}{dy} \right) \\
x &= \frac{1}{\sqrt{2} \alpha} (a + a^+) & y &= \frac{1}{\sqrt{2} \beta} (b + b^+) \\
p_x &= -\frac{i}{\hbar \sqrt{2}} \left( a - a^+ \right) & p_y &= \frac{i}{\hbar \sqrt{2}} \left( b - b^+ \right) \\
H^{(0)} &= -\frac{\hbar^2}{2m} \frac{1}{2\alpha^2} (a - a^+)^2 + \frac{\hbar^2}{2\beta^2} (b - b^+)^2 + \frac{\hbar}{2} \left( \frac{1}{\alpha} (a + a^+)^2 + \frac{1}{\beta} (b + b^+)^2 \right)
\end{align*}$$
Consider the coefficients in front of the brackets:

\[-\frac{\hbar^2 \alpha^2}{2} \cdot \frac{1}{2\mu} = -\frac{\hbar^2}{2\mu} \left( \frac{\hbar w}{\mu} \right)^2 = -\frac{\hbar w}{\mu} \]

\[-\frac{\hbar^2}{2\mu} \left( \frac{\hbar}{\mu} \right) = \frac{\hbar w^2}{\mu} \cdot \frac{1}{\mu} = \frac{\hbar w^2}{\mu^2} \cdot \frac{1}{\mu} = \frac{\hbar w}{\mu^2} \]

so now we have:

\[H^{(0)}_x = -\frac{\hbar w}{\mu} \left( a^+ - a \right)^2 + \frac{\hbar w}{\mu} \left( a^+ + a \right)^2\]

\[= \frac{\hbar w}{\mu} \left( (a^+ + a)^2 - (a^+ - a)^2 \right) = \frac{\hbar w}{\mu} \left( a^+ + a \right) \left( a^+ + a + a^+ + a \right) - a^+ - a\]

\[= \frac{\hbar w}{\mu} \left( 2a^+ a + 1 \right) = \hbar w \left( \frac{\mu + 1}{\mu} \right)
\]

the same for \(H^{(0)}_y = \hbar w \left( \frac{\mu + 1}{\mu} \right)\) or

\[E_{Hx, ny} = \hbar w \left( nx + ny + \frac{1}{2} \right) = \hbar w \left( nx + ny + 1 \right)
\]

The eigenstates are given by \(|nx, ny, 1\rangle \equiv |nx, ny, 1\rangle\).

Note \(E_{Hx, ny}\) is \((nx + ny + 1)\) field degenerate.

E.g., \(101\) and \(110\) will have the same energy.

Let's go to the perturbation term \(H^{(1)} = J_{xy}\)

the secular matrix is:

\[
\begin{vmatrix}
-\hbar w \langle 101 | x y | 10\rangle & -\hbar w \langle 101 | x y | 10\rangle \\
\hbar w \langle 01 | x \rangle | 10\rangle & -\hbar w \langle 01 | x y | 10\rangle & -\hbar w \langle 01 | x y | 10\rangle
\end{vmatrix}
\]

Consider \(\langle 101 | x y | 10 \rangle = \langle 1 \rangle < x | y \rangle 0 \rangle 0 \rangle \).

\[\sin \text{ e.g. } X \times a^+ a \Rightarrow < 110 | a^+ a | 110 > = < 110 | a^+ a | 110 > = < 110 | a^+ a | 110 > \]

\[\cos \text{ e.g. } X \times a^+ a \Rightarrow < 110 | a^+ a | 110 > = < 110 | a^+ a | 110 > = < 110 | a^+ a | 110 > \]
Now let diagonal terms:

\[ \langle 10 \mid \lambda xy \mid 01 \rangle = 1 \langle 11 \times 10 \rangle \langle 01 y 11 \rangle = \frac{1}{2x^2} - \frac{1}{2x^2} = \frac{1}{2x^2} \quad \text{so we get} \]

\[
\begin{vmatrix}
0 & E^{(1)} \\
\frac{1}{2x^2} & 0 - E^{(1)}
\end{vmatrix} = 0 \Rightarrow E^{(1)} = \pm \frac{1}{2x^2}
\]

So after applying this perturbation term we will lift the degeneracy of the states \( |10\rangle \) and \( |10\rangle \)

\[
2\hbar \omega_0 |10\rangle \rightarrow 2\hbar \omega_0 + \frac{\lambda}{2x^2}
\]

\[
2\hbar \omega_0 - \frac{\lambda}{2x^2}
\]

The corresponding new eigen vectors

\[
\begin{pmatrix}

- E^{(1)} & \frac{\lambda}{2x^2} \\
\frac{\lambda}{2x^2} & - E^{(1)}
\end{pmatrix}
\begin{pmatrix}

U_1 \\
U_2
\end{pmatrix} = 0 \quad \text{where} \quad E^{(1)} = \pm \frac{1}{2x^2}
\]

\[
\begin{pmatrix}

- \frac{\lambda}{2x^2} & \frac{\lambda}{2x^2} \\
\frac{\lambda}{2x^2} & - \frac{\lambda}{2x^2}
\end{pmatrix}
\begin{pmatrix}

U_1 \\
U_2
\end{pmatrix} = 0 \Rightarrow U_1 = U_2
\]

so \( \phi_1 = \frac{1}{\sqrt{2}} (|10\rangle + |10\rangle) \) and for \( E^{(1)} = -\frac{\lambda}{2x^2} \) we get \( U_1 = -U_2 \) so \( \phi_2 = \frac{1}{\sqrt{2}} (|10\rangle - |10\rangle) \)

\[ \text{END OF L1} \]

\[ \text{REQUIRED READING: pp 345-350, Ch 13.5} \]

\[ 1^{st} \text{ORDER STARK EFFECT IN HYDROGEN ATOM.} \]

\[ \text{Mahan: pp 170-181, 194-199} \]