Identical Particles

on the way to quantum field theory

Particles are identical when we cannot distinguish them based on any physical property.

→ many-body physics
→ pure quantum phenomena

In classical mechanics we can track any particle individually. Since for quantum we deal with the wave like properties of amplitudes of probability we cannot distinguish them any longer.

\[ e^{-i \frac{2\pi}{k} k} \]

So unless we consider the case when 2 particles are well separated, i.e. without the overlap we cannot talk any longer about the individual objects in QM.

We need new formalism. As usual the best way to deal with this situation is to consider symmetry for the ensemble of the quantum particles.

1. Permutation symmetry

If we consider a system below the threshold for getting it into the excited state we have 4 degrees of freedom, i.e.

\[ \tilde{p}, \tilde{r}, \tilde{\epsilon} \text{ and } \xi \text{ helicity} \]

\[ \text{spin} \]
Consider a system which has \( N \) particles:

\[
\begin{align*}
1 & \equiv \hat{p}_1, \hat{r}_1, \hat{\sigma}_1, \\
2 & \equiv \hat{p}_2, \hat{r}_2, \hat{\sigma}_2, \\
& \text{etc.}
\end{align*}
\]

Hamiltonian can be written as:

\[ H(1, 2, \ldots, N) \] and so is \( \psi(1, 2, \ldots, N) \)

Let's introduce a new operator \( \hat{P}_{ij} \) = the permutation operator

\[
\hat{P}_{ij} \psi(1, 2, \ldots, i, \ldots, j, \ldots, N) = \omega \psi(1, 2, \ldots, i, \ldots, j, \ldots, N)
\]

Do this twice

\[
\hat{P}_{ij}^2 \psi(\ldots, i, \ldots, j, \ldots) = \omega^2 \psi(\ldots, i, \ldots, j, \ldots) \Rightarrow \omega^2 = 1 \text{ or } \omega = \pm 1
\]

Then \( \hat{P}_{ij} \) is a Hermitian operator.

\[
\hat{P}_{ij}^2 = I \Rightarrow \hat{P}_{ij} = \hat{P}_{ij}^{-1}
\]

Also, you can show that \( \hat{P}_{ij} \) is unitary.

\[
\text{i.e. } U^\dagger U^{-1} = I
\]

\[
\hat{P} + \hat{P}^{-1} = \hat{I} \Rightarrow \hat{P} \hat{H} = \hat{H} \hat{P} \text{ and for any symmetric operator } \hat{O} \text{ we get } [\hat{P}, \hat{O}] = 0
\]

If \( \psi(1, \ldots, N) \) is eigenstate of \( \hat{H} \)

then \( \hat{P} \hat{H} \psi = \hat{P} \hat{E} \psi = \hat{E} \hat{P} \psi \) is also an eigenstate.

This is called exchange degeneracy

Also, all observables are the same for \( \psi \) and \( \hat{P} \psi \).

Now we have \( N! \) exchange operators for \( N \) particles, there has to be at least one eigenfunction such as:

\[
\hat{H} \psi = \varepsilon \psi \quad \text{AND} \quad \hat{P}_{ij} \psi = \omega_{ij} \psi
\]

also one can show that \( \hat{P}_{ij} \hat{P}_{jk} = \hat{P}_{jk} \hat{P}_{ij} \)

See Problem 1 ch. 18.
This also means
\[ p_{ij} p_{ik} \psi = w_{ij} w_{ik} \psi \Rightarrow \]
\[ w_{ij} w_{ik} = w_{ik} w_{ij} \]
which gives \( w_{ik} = w_{jk} \)
or \( w_{ij} = w_{ik} = w_{jk} \)
if \( w_{ij} = 1 \), then \( \psi \) is symmetric under interchange of any pair of particles \( \equiv \psi_s (1 \ldots N) \)
if \( w_{ij} = -1 \) \( \psi \) antisymmetric \( \equiv \psi_a (1 \ldots N) \)
\[ p_{ij} \psi_s (1 \ldots i \ldots j \ldots N) = \psi_s (1 \ldots i \ldots j \ldots N) + \psi_s (i \ldots j \ldots i) \]
\[ p_{ij} \psi_a (1 \ldots i \ldots j \ldots N) = -\psi_a (1 \ldots i \ldots j \ldots N) \]
\[ p \psi_s = \psi_s \]
\[ p \psi_a = (-1)^p \psi_a \]
\[ (-1)^p = \begin{cases} 1 & \text{even permutations} \\ -1 & \text{odd permutations} \end{cases} \]
also \( p \psi = e^{i \theta} \psi \) where \( \theta = 0, \pi, 2\pi \)
if \( \theta \) is any \( 0 < \theta < 2\pi \) the particle is called anyon.

E.g. Consider two particles \( N = 2 \)
\( p_{12} \) is the only operator
for 3-particle \( p_{ij} \psi (1, 2, 3) \)
\[ p_{12} \psi (1, 2, 3) = \]
\[ p_{13} \psi (1, 3, 2) = \psi (1, 3, 2) \]
\[ p_{23} \psi (2, 1, 3) = \psi (2, 1, 3) \]
\[ p_{13} p_{23} \psi (1, 2, 3) = \psi (1, 2, 3) \]
\[ p_{13} p_{23} \psi (1, 2, 3) = \psi (1, 2, 3) \]
don't commute!
so there are states where all \( p_{ij} \)'s are not
\( \psi \equiv \text{equal} \).
However, experimentally we know that only \( \theta = 0 \) and \( \theta = \pi \) are
So the Hilbert space is broken into subspaces $H_s$ and $H_a$ and the remaining unphysical $(N! - 2)$ states $H_r$.

Since $P$ commutes with $H$ the symmetric characters cannot be changed over time.

So fermions remain fermions — bosons remain bosons.

Moreover, if we have a system of identical particles they will be described by a uniquely symmetrical wave $\psi$, otherwise the states would mix between symm. and anti-symm.

Spin - statistics theorem.

- A wave function of $N$ identical particles of $\frac{1}{2}$, $n$ where $n$ is odd must have antisymmetric wave function under exchange of any two particles.

- Wave function of any $N$ particles with $\frac{1}{2}$ where $\frac{n}{2}$ is even or 0 must be symmetric.

So there are 2 quantum statistics.

Bose - Einstein $n$ even or 0 $\Rightarrow$ bosons

Fermi - Dirac $n = \frac{1}{2}$, $m = odd. \Rightarrow$ fermions

It works even for composite particles:

- $^3H_e = 2$ protons + neutron

  $2 \cdot \frac{1}{2} + \frac{1}{2} = \frac{3}{2} = \text{fermion}$

- $^4H_e =$ boson = and the subject to B-E condensation $\Rightarrow$ superfluidity
Symmetry of w.f.

Many-body Sch. eqn.
\[ i \hbar \frac{\partial}{\partial t} \psi(1\ldots N) = H(1\ldots N) \psi(1\ldots N) \]

Among all possible solutions we need to construct at least either sym. or antisym. w.f.

Let's see how it's done.

Recall among \( N! \) solutions corresponding the same energy eigenvalue \( \psi_0 \) so we can sum them up and then normalize.

This can be understood that if we have all bunch of \( \psi(1\ldots N) \) one of them into another which is included into this sum.

So the antisym. can be setup as a sum of all \( \psi \) permuted w.f. by means of even interchanges of pairs and substratutions constructed the sum of all the functions by mean of odd number of interchanges.

Example: 2 particles \#(12) Let's assume \( \psi_{(12)} \) is the solution of Sch. eqn.
then \( P_{12} \psi_{(12)} = \psi_{(21)} \) is also a solution
\[
\psi_s = A \left( \psi_{(12)} + \psi_{(21)} \right) \quad \psi_a = B \left( \psi_{(12)} - \psi_{(21)} \right) \\
P \psi_s = A \left( \psi_{(21)} + \psi_{(12)} \right) = \psi_s \quad P \psi_a = A \left( \psi_{(21)} - \psi_{(12)} \right) = -B \psi_{12}
\]
For \( N \) particles

\[
\Psi_s = \mathcal{S}_s \Psi(1 \ldots N, t) = \frac{1}{\sqrt{N!}} \sum_p \Psi(1 \ldots N, t)
\]

\[
\Psi_a = \mathcal{S}_a \Psi(1 \ldots N, t) = \frac{1}{\sqrt{N!}} \sum_p (-1)^p \Psi(1 \ldots N, t)
\]

Ex. 3 particles: \( \Psi(123) \quad N! = 3! = 1 \cdot 2 \cdot 3 = 6 \)

\[
\Psi_s = \frac{1}{\sqrt{6}} (\Psi(123) + \Psi(213) + \Psi(132) + \Psi(321) + \Psi(312) + \Psi(231))
\]

\[
\Psi_a = \frac{1}{\sqrt{6}} (\Psi(123) - \Psi(123) - \Psi(231) - \Psi(132) + \Psi(321) + \Psi(231))
\]

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Note for \( \Psi_s^* \Psi_s \) and \( \Psi_a^* \Psi_a \) the wave function doesn't change under permutation of 2 particles. So any measurable quantity is not sensitive to the particle exchange.

\[
|\Psi_a|^2 = \frac{1}{2} (\Psi(12)^2 + \Psi(21)^2) - \Psi(12)\Psi(21)^* - \Psi(21)\Psi(12)^*
\]

Interaction term!

If particles are well separated, this term = 0 and particles are distinguished.

E.g., for \( kT \) much larger than \( kT \) and low number of particles, # of particles per quantum state is small and we can apply classical statistics.
Pauli Exclusion Principle

No two fermions can be in the same quantum state (i.e. have the same quantum numbers).

It is not possible to solve a many body problem exactly. The way we solve it is to assume particles are non-interacting, 2) include interactions via perturbation theory.

Specifically:

\[ H(\ldots N) = H(1) + \ldots + H(N) \]

\[ \Psi(\ldots N) = \psi(1) \psi(2) \ldots \psi(N) \]

\[ E_0 = E_{\alpha_1} + E_{\alpha_2} + \ldots + E_{\alpha_N} \]

\[ H_0 (\alpha_j) \phi_{\alpha_j} (\alpha_j) = E_\alpha \phi_{\alpha_j} (\alpha_j) \]

The eigenfunction corresponding to \(E_0\) will be a linear combination of \(\Psi(\ldots N)\).

In general \(\Psi_\alpha\) can be written as determinant (Slater determinant):

\[ \Psi_\alpha = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_{\alpha_1} (1) & \phi_{\alpha_2} (2) & \ldots & \phi_{\alpha_N} (N) \\ \phi_{\beta_1} (1) & \phi_{\beta_2} (2) & \ldots & \phi_{\beta_N} (N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\gamma_1} (1) & \phi_{\gamma_2} (2) & \ldots & \phi_{\gamma_N} (N) \end{vmatrix} \]

With \(j\) th particle \(\phi_j (\alpha_j)\) and \(N\) wave functions.

Change of sign comes from the change of sign upon exchange of 2 columns.
By looking at the Slater matrix we can see also if any of two particles say 1 and 2 have the same $\delta_{ij}$ ⇒ the determinant = 0

However for bosons any two or more particles can occupy the same state i.e. the occupation number for bosons 0, 1, 2 ... for fermion it is either 0 or 1.

**SPIN OF 2 ELECTRONS**

\[
H = \frac{1}{2m} \sum_{i=1}^{N} \vec{p}_i^2 + V(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)
\]

has no spin operator or if so is neglected.

⇒ $\Psi(\vec{r}_1, ..., \vec{r}_N)$. If we include spin we need to add the spin eigenfunction $\chi(\vec{r}_1, ..., \vec{r}_N)$ so

$\Psi(1, ..., N) = \Phi(\vec{r}_1, ..., \vec{r}_N) \chi(\vec{r}_1, ..., \vec{r}_N)$

or some sort of linear combo of such product $\Phi \cdot \chi$. This is the 1st approx

for so interaction.

Spin-orbit

We still need to apply the same symmetry argument to the total wave function, but now we consider both $\chi$ and $\Phi$. 

The issue is that we can construct asympt. and sym. parts for $X$ and $\phi$ separately:

\[
\text{sym} \times \text{sym} \rightarrow \text{sym, only for bosons}
\]

\[
\text{asym} \times \text{asym} \rightarrow \text{asym, only possible for fermions}
\]

Let's recall how it works for spins:

For angular moment $j_1, j_2, m_1, m_2$:

\[
j = j_1 + j_2
\]

The same can be applied to spins:

\[
\frac{1}{2} \left( \frac{1}{2} \pm \frac{1}{2} \right) \to \frac{1}{2} \text{ and } \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \right) = 0
\]

The spin part for 2 electrons 1 and 2:

\[
\alpha(1) \beta(2), \alpha(2) \beta(1), \beta(1) \alpha(1), \beta(2) \alpha(2)
\]

Next we can construct 4 commuting operators out of $S$:

\[
S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2
\]

\[
S_1^2 = \frac{1}{2}, S_2^2 = \frac{1}{2}
\]

The common set of eigenstates we label as $(S_0 \phi, \phi S_0)$ where $S = \frac{1}{2} + \frac{1}{2} = 1$, $m = -1, 0, 1$

$\frac{1}{2} - \frac{1}{2} = 0$
The Clabach-Bordon matrix for this representation is:

\[
\begin{align*}
X_s(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \alpha(s) \alpha(s) \\
X_s(0) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow \alpha(s) \beta(s) + \beta(s) \alpha(s) \\
X_s(-1) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \Rightarrow \beta(s) \beta(s) - \alpha(s) \alpha(s) \\
X_a &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \alpha(s) \alpha(s)
\end{align*}
\]

\[
S^2 = (S_1 + S_2)^2 = S(S+1) \quad \text{in units of} \quad \hbar
\]

The eigenstates \(S_2\) are:

\[
\begin{align*}
S_2 = 1 & \Rightarrow m_s = \pm 1 \\
S_2 = 0 & \Rightarrow m_s = 0
\end{align*}
\]

\(X_s(1) = \hbar\) \quad \text{eigenvalue} \quad 2\hbar^2

\(X_a = 0\)

\(X_s(0) = 0\)

\(X_s(-1) = -\hbar\)

\(X_s = \text{TRIPLET STATE AND SYMMETRIC}\)

\(X_a = \text{ANTI-SYMM.}\)

Remember this is only the spin part. The total wave function \(\vec{r} \cdot \vec{r}(\psi)\) still needs to be antisymmetric.

\[\text{See Problem 2 - very instructive on page 453.}\]
Exchange Interaction.

In non-relativistic version of QM, we have no notion that interaction may depend on spin.

Consider a system of 2 electrons.

\[ H(1,2) = K_1(r_1) + K_2(r_2) + V(|r_2 - r_1|) \]

in the c.m. representation

\[ \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2} = \frac{1}{\mu} \]

\[ H(R, r) = H_{rel}(r) + K_{cm}(R) = \frac{1}{2\mu} \rho^2(R) + V(r) \]

\[ + K_{cm}(R) = \frac{1}{2M} \rho_{cm}^2(R) \]

\( \rho_{12} \) commutes with \( H \Rightarrow [\rho_{12}, K_{cm}] = 0 = [\rho_{12}, H_{rel}] = 0 \)

The eigenstates of \( K_{cm} \) and \( H_{rel} \) can be either symmetric or antisymmetric.

E.g. for \( K_{cm} \):

\[ e^{ik_{cm} R} = e^{ik_{cm}(r_1 + r_2)} \]

\[ \rho_{12} e^{ik_{cm}(r_1 + r_2)} = e^{ik_{cm}(r_2 + r_1)} \]

So the overall symmetry actually depends on \( H_{rel} \), and its eigenstate is given by

\[ \phi(r_1, r_2) \cdot \psi(r_1, r_2) \]

1. Assume \( s = 0 \), \( \Rightarrow \) boson \( \Rightarrow \phi(r_1, r_2) \) symmetric now if \( \psi_{hem}(r, \theta, \phi) \) and the exchange is equivalent to \( r \rightarrow -r \) which is the same as \( \psi_{cm}(r_2 - r, \theta, \phi) = (-1)^2 \psi_{hem}(r, \theta, \phi) \)

\( \Rightarrow \) stay symmetric \( \psi_{hem} \) only can have \( \epsilon = \text{even} \)
Now let, say we have $2$-electrons

$s = \frac{1}{2}$

As we discussed above we will have

$\Psi (s, s_2) = \begin{cases} \Psi_s \text{ triplet } \uparrow \uparrow \\ \Psi_s \text{ singlet } \downarrow \uparrow \end{cases}$

Since the total $\psi$ must be antisymm.

we can say that $\Psi_{\text{new}} (r, \Theta, \Phi) = \begin{cases} \text{antisymm. for } \Psi_s \\ \text{symm. for } \Psi_s \end{cases}$

so if $l = \text{even}$ for $s = 0$

$\{ l = \text{odd} \text{ for } s = 1 \}$

overall $l + s = \text{even}$

Now if $\Phi_{a1}$ and $\Phi_{a2}$ are the spatial wave functions

$\uparrow \downarrow: \Phi_s (12) = \frac{1}{\sqrt{2}} \left[ \Phi_{a1} (1) \Phi_{a2} (2) + \Phi_{a2} (1) \Phi_{a1} (2) \right]$

$\uparrow \uparrow: \Phi_a (12) = \frac{1}{\sqrt{2}} \left[ \Phi_{a1} (1) \Phi_{a2} (2) - \Phi_{a2} (1) \Phi_{a1} (2) \right]$

Assume now the electrons are very close to each other:

$\Phi_{a1} (1) \approx \Phi_{a2} (2) \quad \Phi_{a2} (1) \approx \Phi_{a1} (2) \quad \Rightarrow$

$\Phi_a (12) \rightarrow 0$

This means that probability of $2$ electrons (triplet ones) $\uparrow \downarrow$ to come close is very small. Or it may look like they repel each other. This effect is not b/c of their charge but rather from the symmetry consideration of having the overall antisymmetric wave function for fermions.
What about bosons?

in this case \[ \phi_s \equiv \sqrt{2} \phi_{x_1} (1) \phi_{x_2} (2) \]

which is \( X \) over the average value.

Hence 2 non-interacting bosons love to

come together at the same space point if

their eigenstate is symmetric.

So if \((\uparrow \uparrow)\) and \(s_z = 0\) they act like

they attract each other.

This is the idea behind exchange "force."

It looks like the fact of repulsion or

attraction depends oh what spin state our

many-body system is.

This kind of interaction is known as

exchange interaction and very important

in condensed matter and especially

in strongly correlated electronic matter.

This is purely quantum phenomena and

is due to the fact we cannot label

the particles in QM.

Read 18.7 on excited state of He Atom.