HW PHYS 502 - 2018 Solutions

Problem 1

A harmonic oscillator potential is perturbed by the term $\lambda b x^2$. Calculate the first-order and second-order corrections to the energy eigenvalues.

The first-order correction to energy eigenvalues is $E_n^{(1)} = \langle n|H^{(1)}|n\rangle$ where $H^{(1)} = bx^2$. We have the following relations for linear harmonic oscillator:

$a|n\rangle = \sqrt{n}|n-1\rangle, \;\; a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \;\; x = \sqrt{\hbar/(2m\omega)}\,(a+a^\dagger).$

Now

$$\begin{aligned} E_n^{(1)} &= \langle n|H^{(1)}|n\rangle \\ &= \frac{b\hbar}{2m\omega}\langle n|(a+a^\dagger)^2|n\rangle \\ &= \frac{b\hbar}{2m\omega}\langle n|a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a|n\rangle \; . \end{aligned}$$

We note that

$$\begin{aligned} \langle n|a^m|n\rangle &= 0 \; , \quad \langle n|a^{\dagger m}|n\rangle = 0 \; , \\ \langle n|a^m a^{\dagger l}|n\rangle &= 0 \;\text{ if }\; m \neq l \; , \\ \langle n|a^{\dagger m} a^l|n\rangle &= 0 \;\text{ if }\; m \neq l \; . \end{aligned}$$

Hence, in the expression for $E_n^{(1)}$, $\langle n|a^2|n\rangle = 0$ and $\langle n|a^{\dagger 2}|n\rangle = 0$. Then

$$\begin{aligned} E_n^{(1)} &= \frac{b\hbar}{2m\omega}\langle n|aa^\dagger + a^\dagger a|n\rangle \\ &= \frac{b\hbar}{2m\omega}\left[\langle n|a\sqrt{n+1}|n\rangle + \langle n|a^\dagger\sqrt{n}|n-1\rangle\right] \\ &= \frac{b\hbar}{2m\omega}\left[\langle n|n+1|n\rangle + \langle n|n|n\rangle\right] \\ &= \frac{b\hbar}{2m\omega}(2n+1)\langle n|n\rangle \\ &= \frac{b\hbar}{m\omega^2}\left(n+\frac{1}{2}\right)\hbar\omega \; . \end{aligned}$$

Next, consider

$$E_n^{(2)} = \sum_{m\neq n}\left(\frac{\left|H_{nm}^{(1)}\right|^2}{E_n^{(0)} - E_m^{(0)}}\right) \; .$$

We obtain

$$
\begin{aligned}
H_{nm}^{(1)} &= \frac{b\hbar}{2m\omega}\left[\langle n|a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a|m\rangle\right] \\
&= \frac{b\hbar}{2m\omega}\left[\langle n|a\sqrt{m}|m-1\rangle + \langle n|a^\dagger\sqrt{m+1}|m+1\rangle\right. \\
&\qquad \left. + \langle n|a\sqrt{m+1}|m+1\rangle + \langle n|a^\dagger\sqrt{m}|m-1\rangle\right] \\
&= \frac{b\hbar}{2m\omega}\left[\langle n|\sqrt{m}\sqrt{m-1}|m-2\rangle + \langle n|\sqrt{m+1}\sqrt{m+2}|m+2\rangle\right. \\
&\qquad \left. + \langle n|\sqrt{m+1}\sqrt{m+1}|m\rangle + \langle n|\sqrt{m}\sqrt{m}|m\rangle\right] \\
&= \frac{b\hbar}{2m\omega}\left[\sqrt{m(m-1)}\,\delta_{n,m-2} + \sqrt{(m+1)(m+2)}\,\delta_{n,m+2}\right. \\
&\qquad \left. + (m+1)\delta_{n,m} + m\delta_{n,m}\right] .
\end{aligned}
$$

Then

$$
\begin{aligned}
E_n^{(2)} &= \sum_{m\neq n} \frac{\left|H_{nm}^{(1)}\right|^2}{(E_n^{(0)} - E_m^{(0)})} \\
&= \sum_{m\neq n} \frac{\left|H_{nm}^{(1)}\right|^2}{(n-m)\hbar\omega} \\
&= \frac{b^2\hbar^2}{2^2 m^2\omega^2\hbar\omega}\left[\frac{(n+2)(n+1)}{-2} + \frac{(n-1)n}{2}\right] \\
&= -\frac{b^2}{2m^2\omega^4}\left(n+\frac{1}{2}\right)\hbar\omega .
\end{aligned}
$$

Problem 2:

If the Hamiltonian of a particle in a box of length L is subjected to a uniform electric field given by $H^{(1)} = -eEx$ calculate the first-order correction to the energy eigenvalues.

The eigenvalues and the eigenfunctions of the unperturbed system are

$$E_n^{(0)} = \frac{\hbar^2\pi^2 n^2}{4mL^2}\,, \quad n = 1, 2, \cdots\,, \quad \phi_n^{(0)} = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right)\,.$$

The first-order correction $E_n^{(1)}$ is

$$\begin{aligned} E_n^{(1)} &= -\frac{2eE}{L}\int_0^L x\sin^2(n\pi x/L)\,\mathrm{d}x \\ &= -\frac{eE}{L}\int_0^L [x - x\cos(2n\pi x/L)]\,\mathrm{d}x \\ &= -\frac{eE}{L}\left[\frac{L^2}{2} - \frac{L^2}{4n^2\pi^2}\cos(2n\pi x/L)\Big|_0^L\right] \\ &= \frac{eEL}{2}\,. \end{aligned}$$

Problem 3

An electron in the potential $V(x) = \begin{cases} -a/x, & x > 0 \\ \infty, & x \le 0 \end{cases}$ is in its ground state characterized by the eigenfunction $\phi_0 = Nx\mathrm{e}^{-\beta x}$. The particle is subjected to an applied electric field E_e in the x-direction. Calculate the first-order correction to ground state energy eigenvalue.

First, calculate β. Substituting the solution in the Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi_{xx} - \frac{a}{x}\psi = E\psi$$

we get

$$-\frac{\hbar^2}{2m}\left[-2\beta+\beta^2 x\right]-a=Ex\ .$$

Equating the coefficients of x^0 and x equal to zero separately we get

$$E=-\frac{ma^2}{2\hbar^2}\ , \quad \beta=\frac{am}{\hbar^2}\ .$$

Next, from the normalization condition $1=N^2\int_0^\infty x^2\mathrm{e}^{-2\beta x}\,\mathrm{d}x$ we find $N=2\beta^{3/2}$.

Next, the first-order correction to the ground state energy can be calculated from $E_0^{(1)}=\langle H^{(1)}\rangle$. The perturbation $H^{(1)}$ due to the applied electric field is $eE_{\mathrm{e}}x$. Then

$$\begin{aligned}
E_0^{(1)} &= N^2eE_{\mathrm{e}}\int_0^\infty x^3\mathrm{e}^{-2\beta x}\,\mathrm{d}x \\
&= \frac{N^2eE_{\mathrm{e}}}{16\beta^4}\int_0^\infty y^3\mathrm{e}^{-y}\,\mathrm{d}y \\
&= \frac{N^2eE_{\mathrm{e}}}{16\beta^4}\left[-y^3\mathrm{e}^{-y}\Big|_0^\infty+3\int_0^\infty y^2\mathrm{e}^{-y}\,\mathrm{d}y\right] \\
&= \frac{3N^2eE_{\mathrm{e}}}{8\beta^4} \\
&= \frac{3eE_{\mathrm{e}}}{2\beta} \\
&= \frac{3eE_{\mathrm{e}}\hbar^2}{2ma}\ .
\end{aligned}$$