Collision between particles

Lecture #6

(identical) Also read Feynman's Lectures Ch. 4-1

\[ \text{detector} \]

\[ \text{this is a center of mass picture.} \]

\[ \text{(or) } + \]

\[ \text{D1} \]

\[ \text{D2} \]

\[ \text{1} \]

\[ \text{2} \]

Some have two independent channels. And we cannot decide which way we scatter particles 1 and 2, and we have to sum up the amplitudes for both events.

Note, classically we would have a different cross-section which is the sum of \( f \), meaning we adding up probabilities and not amplitudes.

\[ \frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 \]

and \( f \) is defined from the scattered w.f.

\[ \psi_r \rightarrow \psi_{\theta, \phi} = e^{i\mathbf{k} \cdot \mathbf{r}} + f(\theta, \phi) \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{r} \]

Now recall the product for bosons and fermions is very different.

1. Let's assume we scatter 2 spin-1 bosons

\[ r_1 \rightarrow r_2 \rightarrow 1r_1 = r_1 - r_2 \Rightarrow r_2 \rightarrow -r_1 \]

and in the polar coordinates means that \( \theta, \phi \)
\[ Y_\ell^m (r \rightarrow +\infty) = e^{ikr} + e^{-ikr} + \left[ f(\theta, \phi) + f(\pi-\theta, \pi+\phi) \right] e^{ikr} \]

\[ \frac{d\sigma}{d\Omega} = \left| f(\theta, \phi) + f(\pi-\theta, \pi+\phi) \right|^2 \]

\[ = \left| f(\theta, \pi) \right|^2 + \left| f(\theta, \phi+\pi) \right|^2 + 2 \text{Re} \left[ f(\theta, \phi) \right] \quad \text{this is extra compared to classical scattering.} \]

\[ f^* (\pi-\theta, \phi+\pi) \]

If the potential is independent of \( \phi \) (e.g. central potential) \( \Rightarrow \)

\[ \frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 + 2 \text{Re} \left[ f(\theta). f(\pi-\theta) \right] \]

Note if \( \theta = \frac{\pi}{2} \rightarrow \frac{d\sigma}{d\Omega} = 4 \left| f(\theta) \right|^2 \]

\[ \uparrow \quad \text{the symmetry angle in the c.o.f. } \]

\[ \uparrow \quad \text{so } 1 + 1 = 4 \]

Moreover recall in the phase shift analysis lecture to be symmetric \( \theta \rightarrow \pi-\theta \)

\[ \left( P_e(-x) = (-1)^x P_e(x) \right) \quad \text{It can contain only even } \ell \]

(2) Scattering of 2 fermions spin = \( \frac{1}{2} \)

The total wave func. must be antisymmetric.
The spin part of the w.f. can be symmetric or antisymmetric. Then the spectral part must be symmetric for $\uparrow \uparrow$ and antisymmetric for $\uparrow \downarrow$. Assume the potential is central and spin independent. Then

$$\begin{align*}
fs &= f(\theta) + f(\theta - \pi), \\
fa &= f(\theta) - f(\theta - \pi)
\end{align*}$$

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 + 2 \text{Re} \left[ f(\theta) f^*(\pi - \theta) \right]$$

and

$$\left( \frac{d\sigma}{d\Omega} \right)_{\uparrow \downarrow} = \left| f(\theta) \right|^2 + \left| f(\pi - \theta) \right|^2 - 2 \text{Re} \left[ f(\theta) f^*(\pi - \theta) \right]$$

Assume that incoming fermions are up-polarized. E.g.,

<table>
<thead>
<tr>
<th>Fraction</th>
<th>$S\uparrow$</th>
<th>$S\downarrow$</th>
<th>Spin $\uparrow$ in $D_1$</th>
<th>Spin $\uparrow$ in $D_2$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>$\uparrow$</td>
<td>$\frac{1}{2} \left</td>
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<tr>
<td>$\frac{1}{2}$</td>
<td>$\downarrow$</td>
<td>$\uparrow$</td>
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<td>$\downarrow$</td>
<td>$\frac{1}{2} \left</td>
</tr>
</tbody>
</table>

Total: $= \frac{1}{2} \int \left[ \left| f(\theta) - f(\pi - \theta) \right|^2 + \left| f(\theta - \pi) \right|^2 + \frac{1}{2} \left| f(\theta - \pi) \right|^2 + \frac{1}{2} \left| f(\theta) \right|^2 \right] d\theta$
Total cross section = unpolarized \[ \frac{1}{4} \left( \frac{d\sigma}{d\Omega} \right)_{uu} + \frac{3}{4} \left( \frac{d\sigma}{d\Omega} \right)_{un} \]
\[ = |f(0)|^2 + 1 |f(\pi)|^2 - \frac{1}{2} \int (f(\theta) f^*(\pi-\theta))^2 \]

Compared to bosons, the cross section is a factor of 4 less.

Also:
At \[ \theta = \frac{\pi}{2} \]

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{fermions}} \Theta = \frac{\pi}{2} = |f(\Theta = \frac{\pi}{2})|^2 \]

For bosons \[ \Theta = \frac{\pi}{2} \]

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{bosons}} \Theta = \frac{\pi}{2} = 2 |f(\Theta = \frac{\pi}{2})|^2 \]
Occupation number representation

This idea is very useful for many body theory or quantum field theory.

1. Particle in the box of size $L$.

   \[ \text{let's set } \hbar = 1, \quad p = -i \frac{\partial}{\partial x}; \quad \psi(x) = \frac{1}{\sqrt{2}} e^{i p x} \]

   \[ p \psi(x) = -i \frac{\partial \psi}{\partial x} = p \psi(x) \]

   \[ \text{if } \psi(x) = \psi(x + L), \quad e^{i p x} = e^{i p (x + L)} \Rightarrow p = \frac{2\pi n_m}{L} \]

   **NEW NOTATION**

   on to a multi-particle state (e.g. bosons):

   \[ \left| p_1 p_2 \right> = (p_1 + p_2) \left| p_1 p_2 \right> \]

   \[ H \left( p_1 p_2 \right) = (E_1 + E_2) \left| p_1 p_2 \right> \]

   What if I have 2 particles in $p_3$? $E_{p_3} = 2E_{p_3}$

   double of a single particle energy

   In general:

   \[ \sum \frac{n_{p_m} E_{p_m}}{m}, \quad n_{p_m} \text{ is the total number of particles in the state } p_m \]

   In QFT instead of listing what particle is in what state we can say two particles are in $p_1$, 1 particle in $p_2$, etc.

   so we just specify how many in what state $p_1, \ldots, p_n$

   \[ \text{e.g. } 12100... \]

   number of particles in this momentum state, is called **occupation number representation**

   \[ \text{e.g. } |12100...> \]

   \[ 19191919 > \]

   \[ 12071292 > 1027 \]

   \[ 11119 > \quad \text{etc.} \]

   \[ 1307 \]
What we can do on this state is

\[ \langle \mathbf{H} | \mathbf{n}_L, \mathbf{n}_R \cdots \rangle = \left[ \sum_m \hbar p_m E_{p_m} \right] \mathbf{n}_L, \mathbf{n}_R \cdots \]

simply we find out how many particles
in that state & energy of that state.

Big Q: Why do we care?

Recall in harmonic oscillator:

\[ E_n = (n+\frac{1}{2})\hbar \omega \quad \text{or} \quad E_n = n \hbar \omega \]

so in the oscillator we have \( n \) quanta,
and the energy between states is equally
spaced. Independent

Now imagine \( N \) oscillators each labeled by \( k \), and the spacing is \( \hbar \omega_k \).
The total \( E = \sum_k E_{k} \) so the \( k \)th oscillator,

\[ E_k = \sum_{n=0}^{\infty} (n+\frac{1}{2}) \hbar \omega_k \cdot n_k \]

\[ E_k = \sum_{n=0}^{\infty} n \hbar \omega_k \cdot n_k \]

e.g. \( k = 3 \) \( \hbar \omega_3 \) oscillator has \( n_3 \) quanta in it
and contributes to the energy \( \hbar \omega_3 \cdot n_3 \).

In general:

\[ E = \sum_m \hbar p_m E_{p_m} \quad \text{so we say,} \]

the momentum state \( p_m \) has \( h p_m \) particles in it
and contributes \( h p_m E_{p_m} \) energy.

So it looks like we can think of a general
system as analogous to oscillators.

**Summary:**

<table>
<thead>
<tr>
<th>Quanta in oscillators</th>
<th>Particles in momentum states</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )th oscillator</td>
<td>( m )th momentum mode ( p_m )</td>
</tr>
<tr>
<td>( E = \sum_{k=1}^{N} n_k \hbar \omega_k )</td>
<td>( E = \sum_{m=1}^{N} n_{p_m} E_{p_m} ).</td>
</tr>
</tbody>
</table>
VERY IMPORTANT STEP

REPLACE STATE VECTOR WITH AN OPERATOR!

What's next: can we remove the notion of state vectors at all?

\[ |n_1, n_2, ..., n_N\rangle = \prod_{k} \frac{1}{\sqrt{n_k!}} (a_k^+)^{n_k} |0\rangle \]

so we retain only one very special state \( |0\rangle \)

\[ |n_1, n_2, ..., n_N\rangle = \frac{1}{\sqrt{n_1! n_2! ... n_N!}} (a_1^+)^{n_1} (a_2^+)^{n_2} ... (a_N^+)^{n_N} |0, 0, 0\rangle \]

The general state of the harmonic oscillator:

\[ |n_1, n_2, ..., n_N\rangle = \prod_{k} \frac{1}{\sqrt{n_k!}} (a_k^+)^{n_k} |0\rangle \]

we create a particle with momentum \( p_i \)

\[ |2, 1, 0, 0, 0, 0\rangle = \left[ \frac{1}{\sqrt{2!}} (a_1^+) \right] ^2 \left[ \frac{1}{\sqrt{1!}} (a_2^+) \right] |0\rangle \]

so we can think of this situation as \( a_2^+ \) creating a particle with momentum \( p_2 \)

But we need to think of \( |p_2\rangle \)

In distinguishability & symmetry

What I want to do is to repeat the same consideration about sym. and anti-sym. argument for bosons and fermions.

E.g.: let's add another particle into the vacuum:

\[ \begin{align*}
    a_{p_1}^+ |10\rangle &= |110\rangle \\
    a_{p_2}^+ |10\rangle &= |101\rangle \\
    a_{p_1}^+ a_{p_2}^+ |0\rangle &= |111\rangle \\
    a_{p_1}^+ a_{p_2}^+ |0\rangle &= |111\rangle \\
    a_{p_1}^+ a_{p_2}^+ &= \lambda a_{p_2}^+ a_{p_1} \\
    \lambda &= \pm 1
\end{align*} \]
As before we select: 
\[ a^+_{p_2} a^+_{p_1} = a^+_{p_1} a^+_{p_2} \rightarrow [ ] = 0 \]
\[ [a_i^+ a_j] = \delta_{ij} \]

Those commutation rules are the same as for oscillators.

The many particle state of bosons:
\[ |n_1, n_2, \ldots > = \prod_m \frac{1}{(h_{p_m}!)^{1/2}} (a^+_{p_m})^{n_{p_m}} 10 > \]

\[ a^+_{p_1} a^+_{p_2} 10 > = a^+_{p_2} a^+_{p_1} 10 > = 11_{p_1} 1_{p_2} > \]

In general:
\[ \{ a_i^+ 1, n_1, n_2, \ldots > = \sqrt{k_{i+1}} 1, n_1, n_2, \ldots > \]
\[ a_i^+ 1, n_1, n_2, \ldots > = \sqrt{k_i} 1, \ldots n_i, \ldots > \]

**Fermions**

**Case 2: \( \lambda = -1 \)**

\[ \{ c_i^+ c_j^+ \} \equiv c_i^+ c_j^+ + c_j^+ c_i^+ = 0 \]

If I set \( i \neq j \rightarrow c_i^+ c_i = 0 \) (no way)

\( \text{For fermions} \quad \text{anti-commutator} \)

\[ c_i^+ |n_1, n_2, \ldots > = (-1)^{Z_i} \sqrt{1-n_i} |n_1, \ldots, n_i+1, \ldots > \]

\[ c_i^+ |1, \ldots n_i, \ldots > = (-1)^{Z_i} \sqrt{k_i} |1, \ldots, n_i-1, \ldots > \]

\[ (-1)^{Z_i} \equiv (-1)^{n_i+k_i+k_{i-1} + \ldots + k_1} \]

**Pauli exclusion principle.**

Check that \( n^i c_i^+ c_i \) works!
The continuous limit

1) \( \delta_{ij} \rightarrow \delta^3(p) \)

As the size of the system goes up, spacing in \( p \) goes very much down.

\[ [a_\mathbf{p}^+ a_\mathbf{q}] = \delta^3(p-q) \quad \text{and} \]

\[ H = \int d^3\mathbf{p} \, \varepsilon \mathbf{p} a_\mathbf{p}^+ a_\mathbf{p} \]

E.g., for a single-particle state

\[ \langle \Psi | \mathbf{p}' \rangle = \langle 0 | a_\mathbf{p}^+ 10 \rangle \]

\[ \langle \Psi | \mathbf{p} \rangle = \langle 0 | [\delta^3(p-p') + a_\mathbf{p}^+ a_\mathbf{p}] 10 \rangle = \]

\[ = \langle 0 | \delta^3(p-p') 10 \rangle = \delta^3(p-p') \]

So it works and we can rewrite both operators and states in terms of the number of particles with momentum \( p \) and the very special state \( 10 \).

**Summary:**

- The occupation number representation describes states by listing the number of identical particles in each quantum state.
- We focus on the vacuum state \( |0\rangle \) and then construct many-particle states by acting on \( |0\rangle \) with creation operators.
- To obey the symmetries of many-particle mechanics, bosons are described by commuting operators and fermions are described by anticommuting operators.