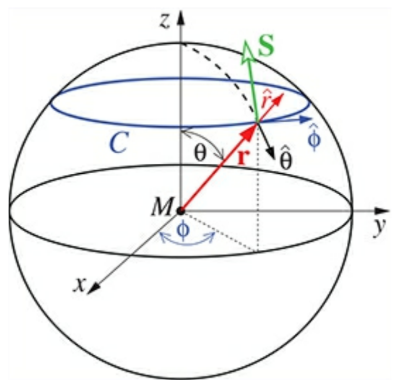
HOMEWORK P502

BERRY'S PHASE FOR SPIN IN A MAGNETIC MONOPOLE FIELD

A useful model for illustrating the ideas of Berry curvature and Berry's phase consists of a spin interacting with a magnetic monopole (referred to as the 'Dirac monopole') field. Consider a spin $\vec{S} = \vec{S}\cdot\vec{\sigma}$ situated at $\vec{r}$, where $\vec{\sigma}$ is the usual vector with components the Pauli matrices, $\sigma = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$. The Dirac monopole of magnitude M situated at the origin produces a magnetic field at the position $\vec{r}$ given by: $\vec{B} = \frac{M}{r^3}\vec{r} = \frac{M}{r^2}\hat{r}$, with $\hat{r}$ the unit vector in the radial direction and $|\vec{r}| = r$ the distance from the origin which we will take to be fixed. See Figure below.



The interaction hamiltonian is: $H = \vec{B}\cdot\vec{S} = \frac{M}{r^3}\vec{r}\cdot(S\vec{\sigma}) = \vec{h}\cdot\vec{\sigma}$, here $\vec{h} = \epsilon_0\hat{r}$ and $\epsilon_0 = \frac{MS}{r^2}$, with ϵ_0 a constant expressed in terms of the parameters M, S and r of the model. Although this system is hypothetical it serves as a simple prototypical model because hamiltonians of other physically motivated models are completely equivalent to it.

1. Write components of the hamiltonian H in spherical coordinates.
2. Find two eigenstate and eigenvalues.
3. Check that those eigenstates are orthogonal and normalized.
4. What are the eigenstates at the north $\theta = 0$ and π - the south pole?
5. We can now imagine that the spin is moved around a horizontal (constant-latitude) circle, which means θ is fixed and ϕ is between $[0, 2\pi]$. What is Berry's phase for this change in position? For this first calculate the Berry connection components $A_\theta(\theta,\phi)$ and $A_\phi(\theta,\phi)$.

6. From those calculate the Berry curvature $\Omega_{\theta,\phi}(\theta, \phi)$
7. Calculate the Berry phase as an integral. For this use the Berry connection integrated over the closed path around the horizontal circle C with the variable of integration denoted by $\omega' = (\theta, \phi')$, so that the path C is described by $\omega' = \hat{\phi} d\phi'$ and thus $\gamma(\theta) = \int_C A(\omega') d\omega' = \int_0^{2\pi} A(\theta, \phi) d\phi'$
8. Find the integral of the Berry curvature over the entire sphere and let $\theta = \pi$.
9. Calculate the Chern number for this problem dividing the number from 8. by 2π.