Lecture 8

Spin of Dirac Particles

In the Heisenberg picture we get

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

For a free Dirac particle $H = c\,\alpha\cdot p + \beta mc^2$

Lets introduce a time evolution for this particle with an angular momentum $\hat{L}$:

$$-i\hbar \frac{\partial}{\partial t}\vec{L} = [H, \vec{L}]$$

$$\vec{L} \rightarrow (L_x\ L_y\ L_z)$$

Consider only one component; say L_3

$$-i\hbar \frac{\partial L_3}{\partial t} = [H, L_3] = \Big[\sum_{k=1}^{3} c\,\alpha_k p_k + \beta mc^2, \underbrace{x_1 p_2 - x_2 p_1}_{=L_3}\Big]$$

recal α_k and β are matrices. and

$$[\beta mc^2, x_1 p_2 - x_2 p] = 0 \leftarrow \text{can you tell why?}$$

$$-i\hbar \frac{\partial L_3}{\partial t} = \Big[\sum c\,\alpha_k p_k, x_1 p_2\Big] - \Big[\sum c\,\alpha_k p_k, x_2 p_1\Big] = -i\hbar c \sum \alpha_k \delta_{k1} p_2 + i\hbar c \sum \alpha_k \delta_{k2} p_1$$

$$= -i\hbar c\,(\alpha_1 p_2 - \alpha_2 p_1) = -i\hbar\,(\alpha \times \vec{p})_3\text{-d component}$$

Here I used $[p, x] = -i\hbar$ $[p_i\, p_j] = 0$ ②

the same we can derive for L_2 and L_1

in other words:

$$\frac{d\vec{L}}{dt} = c(\alpha \times p) \neq 0$$

so L is not a constant of motion

We should construct a new operator to cancel out $-c(\alpha \times p)$ and make it a good operator. i.e,

$L + A$ = is conserved if $\frac{dA}{dt} = -c(\alpha \times p)$

We now can demonstrate that

$$\bar{A} = \frac{\hbar}{2} \cdot \bar{\sigma}^* \quad \text{where} \quad \sigma^* = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

For this purpose we calculate

$$-i\hbar \frac{d\sigma_3^*}{dt} = [H, \sigma_3^*] =$$

$$= \left[\sum c\,\alpha_k \cdot p_k + \beta mc^2, \sigma_3^*\right] = 2i\,(\alpha_1 p_2 - p_1 \alpha_2)$$

again we use $[\beta mc^2, \sigma_3^*] = 0$

scalar

$[\alpha_3, \sigma_3^*] = 0$

$\sigma_1^* \sigma_3^* = -\sigma_3^* \sigma_1^*$

and $\sigma_1^* \sigma_3^* = -i\sigma_2^*$ (show this with Pauli matrices)

that is :
$$\frac{\hbar}{2}\frac{d\sigma_3^*}{dt} = -c(\alpha\times p)_3$$
the same for σ_1^* and σ_2^*, in short
$$\frac{d}{dt}\frac{\hbar}{2}\vec{\sigma}^* = -c(\alpha\times\bar{p})$$
Combining this with the expression for $\vec{L}$
$$\frac{d\bar{L}}{dt} = c(\alpha\times\bar{p}) \Rightarrow \frac{d}{dt}\Big(\bar{L} + \underbrace{\frac{\hbar}{2}\bar{\sigma}^*}_{\text{spin angular moment}}\Big) = 0$$
Thus a new quantity :
$$\bar{J} = \bar{L} + \frac{\hbar}{2}\bar{\sigma}^*$$
is conserved $\frac{d\bar{J}}{dt} = 0$

and the Dirac particle has spin $\frac{\hbar}{2}\sigma^*$

Dirac PARTICLE IN A POTENTIAL

To solve a problem of a Dirac particle in the potential V we 1st modify the equation to include the potential.

Intuitively we can write down:

$$H = c\alpha\cdot\bar{p} + \beta mc^2 + V =$$

$$= -i\hbar c \frac{d}{dz}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} mc^2 + V(z)$$

where we selecte $\alpha \equiv \alpha_3$

for $H\psi = E\psi \Rightarrow$

$$\left[i\hbar c \frac{\partial}{\partial z}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + mc^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + V \right]\begin{pmatrix} v \\ w \end{pmatrix} =$$

$$= E\begin{pmatrix} v \\ w \end{pmatrix}$$ $\Rightarrow$ multiplying those matrices

$$\begin{cases} -i\hbar c \underbrace{w'}_{\frac{dw}{dz}} + v(mc^2 + V) = Ev \\ \hbar c v' - w(mc^2 + V) = Ew \end{cases}$$

Recall $\overset{E_1}{\Psi_1} = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$ and $\overset{E_2}{\Psi_2} = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}$

those are two bound state Solutions

So we have 4 eqns for those components

(5)

$$\begin{cases} -\hbar c\, w_1' + (mc^2+V)\, v_1 = E_1 w_1 & \cdot\, (-) v_2^* \\ \hbar c\, v_1' - (mc^2+V)\, w_1 = E_1 v_1 & \cdot\, w_2^* \\ -\hbar c\, w_2' + (mc^2+V)\, v_2 = E_2 v_2 & \cdot\, (-) w_1^* \\ \hbar c\, v_2' - (mc^2+V)\, w_2 = E_2 w_2 & \cdot\, v_1^* \end{cases}$$

$$\begin{cases} \hbar c\, (v_1 w_2' - w_1' v_2) = (E_1 - E_2)\, v_1 v_2 \\ \hbar c\, (v_1' w_2 - w_1 v_2') = (E_1 - E_2)\, w_1 w_2 \end{cases}$$

$$\hbar c \frac{d}{dz}\left(v_1 w_2 - w_1 v_2\right) = (E_1 - E_2)\, \psi_1^* \psi_2$$

(1) if $E_1 = E_2 \Rightarrow \frac{d}{dz}\left(v_1 W_2 - v_2 w_1\right) = 0$

$\downarrow$ const.

We assume that v and $w \to 0$ if $z \to \infty$

so the $\underline{\text{constant} = 0}$

that is: $\frac{v_1}{w_1} = \frac{v_2}{w_2}$, Returning g.

or $\frac{v_1}{v_2} = \frac{w_1}{w_2} \Rightarrow \psi_1 \propto \psi_2$

meaning they represent the same state

meaning also they are NON-DEGENERATE

and also $\underbrace{\langle \psi_1 | \psi_2 \rangle}_{=0} = \int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dz =$

(6)

recall $\hbar c \frac{d}{dz}(v_1 w_2 - w_1 v_2) = (E_1 - E_2)\psi_1^* \psi_2$

$$\psi_1 \psi_2^* = \frac{\hbar c}{E_1 - E_2} \frac{d}{dz}(v_1 w_2 - w_1 v_2) \Rightarrow \int \ldots$$

$$\frac{1}{E_1 - E_2}(v_1 w_2 - w_1 v_2)\Big|_{-\infty}^{\infty} = 0$$

or ψ_1 and ψ_2 are orthogonal

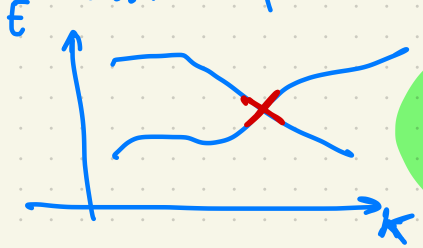
\+ non-deneracy

This means that level crossing cannot occur!

When the potential varies smoothly the w.f. also vary smoothly

x ← some parameter e.g. k momentum

and for the levels to cross

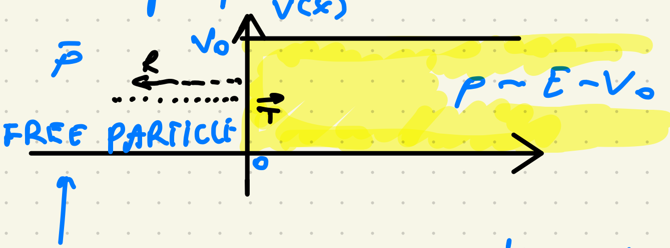


this requiare a SINGULAR behavior in $\langle \psi_1 | \psi_2 \rangle$ when $E_1 = E_2$

Klein "Paradox"

(verified in graphene)

Consider a scattering process by the step potential



1st we construct the plane wave solution of Dirac's free electron in 1D:

$$E\psi = (c\,\alpha_x p_x + \beta mc^2)\psi \quad \text{or}$$

$$i\hbar\frac{\partial\psi}{\partial t} + i\hbar c\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\frac{\partial\psi}{\partial x} - mc^2\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\psi = 0$$

or in the x-component form:

$$i\hbar\frac{\partial\psi_1}{\partial t} + i\hbar c\frac{\partial\psi_4}{\partial x} - mc^2\psi_1 = 0$$

$$i\hbar\frac{\partial\psi_2}{\partial t} + i\hbar c\frac{\partial\psi_3}{\partial x} - mc^2\psi_2$$

$$i\hbar\frac{\partial\psi_3}{\partial t} + i\hbar c\frac{\partial\psi_2}{\partial x} - mc^2\psi_3 = 0$$

$$i\hbar\frac{\partial\psi_4}{\partial t} + i\hbar c\frac{\partial\psi_1}{\partial x} - mc^2\psi_4 =$$

Note: ψ_1 couples only to ψ_3 and ψ_2 to ψ_4.

b/c of this let's into a 2-component (8)
spinor, with $\psi_u = \psi_1$ or ψ_2

$\psi_e = \psi_3$ or ψ_4

for stationary states we get:

$$\begin{cases} \text{V } i\hbar c\,\psi_1' + (E + mc^2)\psi_4 = 0 & \text{V } i\hbar c\,\psi_2' + (E + mc^2)\psi_3 = 0 \\ \text{VV } i\hbar c\,\psi_3' + (E - mc^2)\psi_2 = 0 & \text{VV } i\hbar c\,\psi_4' + (E - mc^2)\psi_1 = 0 \end{cases}$$

in terms of ψ_e and ψ_u we can rewrite it as a single equation

$$\begin{cases} i\hbar c\,\psi_u' + (E + mc^2)\psi_e = 0 \\ i\hbar c\,\psi_e' + (E - mc^2)\psi_e = 0 \end{cases}$$

rename $\psi_u = v$ and $\psi_e = w$

$$\begin{cases} i\hbar c\,v' + (E + mc^2)\,w = 0 & \text{V} \\ i\hbar c\,w' + (E - mc^2)\,v = 0 & \text{VV} \end{cases}$$

$\frac{d}{dx}$ V and using w' from VV

$$i\hbar c\,v'' + (E + mc^2)\,w' = 0$$

$$w' = \frac{v\,(E - mc^2)}{i\hbar}$$

$$v'' + \frac{p^2}{\hbar^2} v = 0 \quad \text{where } p^2 = \frac{1}{c^2}(E^2 - m^2c^4)$$

and putting v'' into $w = -\frac{i\hbar c}{E + mc^2} \cdot v'$

For $v'' + \frac{p^2}{\hbar^2} v = 0$

- $v = Ae^{ipx/\hbar} + Be^{-ipx/\hbar}$
- and $w = \frac{-(i\hbar c)}{E + mc^2} \cdot v'$

Based on this we can write down:

$$v_<(x) = A\left(e^{ipx/\hbar} + R e^{-ipx/\hbar}\right)$$

$$w_<(x) = A\left(a e^{ipx} - aR e^{ipx/\hbar}\right)$$

$$a \equiv cp/(E + mc^2)$$

From solution for $<$ we get the solution for $>$ by replacing E by $E - V_0$

The w.f. for $x<0$ is:

$$\psi_< = \begin{pmatrix} v_< \\ u_< \end{pmatrix} = A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} e^{ipx/\hbar} + R\begin{pmatrix} 1 \\ -a \end{pmatrix} e^{ipx/\hbar}\right] \equiv A\left[u_+ e^{ipx/\hbar} + R\, u_- e^{-ipx/\hbar}\right]$$

where $u_\pm = \begin{pmatrix} 1 \\ \pm a \end{pmatrix}$

for $x>0$ there is no reflected wave

so $\psi_> = \begin{pmatrix} v_> \\ w_> \end{pmatrix} = D\,\bar{u}\, e^{ipx/\hbar}$

here $\bar{u} = \begin{pmatrix} 1 \\ b \end{pmatrix}$ $\quad \bar{p} = p(E - V_0) = \frac{1}{c}[(E - V_0)^2 - m^2c^4]^{1/2}$

and $b = \dfrac{c\bar{p}}{(E - V_0 + mc^2)}$

As usual to determine those constants A, D and R we use

$$\begin{cases} \psi_<(0) = \psi_>(0) \\ \psi'_<(0) = \psi'_>(0) \end{cases}$$

for ψ: $D\bar{u}\, \cancel{e^{ipx/\hbar}} = A\left[u_+ \cancel{e^{ipx/\hbar}} + R u_- \cancel{e^{-ipx/\hbar}}\right.$ $\downarrow_{x=0}$

ψ': . . . . . .

$A(U_+ + RU_-) = D\bar{u} \Rightarrow$

$$A\left[\begin{pmatrix} 1 \\ a \end{pmatrix} + R\begin{pmatrix} 1 \\ -a \end{pmatrix}\right] = D\begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\Rightarrow \begin{cases} A(1 + R) = D \\ Aa(1 - R) = bD \end{cases} \Rightarrow R = \frac{a - b}{b + a}$$

$$T = \frac{2a}{a + b}$$

Behavior of the w. f. depends on V_0

Consider ① $E > V_0 + mc^2$

② $V_0 - mc^2 < E < V_0 + mc^2$

③ $E < V_0 - mc^2$

① $E > V_0 + mc^2 \Rightarrow$

$$E^2 > m^2c^4 + \underbrace{V_0^2}_{>0} + \underbrace{2mc^2V_0}_{>0} \Rightarrow$$

$$E^2 - m^2c^4 > 0$$

$p^2c^2 = E^2 - m^2c^4$ also $> 0 \Rightarrow$

<u>p is real</u>

Since $E - V_0 > mc^2$ $\quad (E - V_0)^2 > m^2c^4$ or

$$\bar{p} = \left[(E - V_0)^2 - m^2c^4\right]^{1/2} > 0$$

$\Rightarrow$ and $\bar{p}$ <u>is real</u>

as such if

$$E > V_0 + mc^2$$

ALL LIKE NRQM!

$x < 0$: $\underset{\text{incoming}}{e^{ipx}} + \underset{\text{reflected}}{e^{-ipx}}$ $\qquad x > 0$ $\underset{\text{transmitted}}{e^{ipx}}$

2 $V_0 - mc^2 < E < V_0 + mc^2$

in this case $(E - V_0)^2 < m^2c^4$

or $\bar{p} = \left[(E - V_0)^2 - m^2c^4 \right]^{1/2}$

is IMAGINARY

Then we have for:

$x < 0$ incoming + reflected

$x > 0$ exponentially decaying wave

LIKE IN NRQM.

Finally

3 $E < V_0 - mc^2$

$E - V_0 < -mc^2 \Rightarrow$

$(E - V_0) < 0$ and

$(E - V_0)^2 > m^2c^4 \Rightarrow$

$\bar{p}$ is Real !

Meaning that we have oscillatory behaviour after the barrier!!??

Recall for NRQM no such solution is possible.

But wait as $E_0 - V_0 + mc^2 < 0$

and thus b is negative

$$\left(b = \frac{c \bar{p} > 0}{\underbrace{(E - V_0 + mc^2)}_{< 0}} \Rightarrow b < 0 \right)$$

we can write dow

$$\boxed{|R| = \left| \frac{a-b}{a+b} \right| > 1}$$

Klein paradox: The amplitude of the reflected waive is LARGER than incoming one.

or

More particles gets reflected than arrived

Also one can show tha even for $V_0 \to +\infty$

$$T = \frac{2p}{E+p} \neq 0 \;!!\,??$$

Dirac electron in the field

Electric field is described by

$A_\mu = \left(\frac{i\varphi}{c}, \bar{A}\right)$ and Dirac eqn is simply is modified as $\bar{p} \to \bar{p} - \frac{e}{c}\bar{A}$

$E \to E - e\varphi$

$$\begin{cases} (\overset{E-e\varphi}{E} - mc^2)v = c(\sigma \cdot p)w \\ (\underset{E-e\varphi}{E} + mc^2)w = c(\sigma \cdot p)v \end{cases}$$

Now we are interested in positive solutions in the form

$$E = \underset{>0}{\mathcal{E}} + mc^2$$

$$(2mc^2 + \overset{0}{\mathcal{E}} - \overset{0}{e\varphi})w = c\sigma \cdot \left(p - \frac{e}{c}A\right)v$$

in a weak field $\mathcal{E}$ and $e\varphi$ are small

so $2mc^2 w \cong c\sigma \cdot \left(p - \frac{e}{c}A\right)v \Rightarrow$

$$w \simeq \frac{1}{2mc}\sigma \cdot \left(\bar{p} - \frac{e}{c}\bar{A}\right)v$$

$$\underset{\uparrow\, E = \mathcal{E} + mc^2}{(E - e\varphi - mc^2)}v \cong \frac{c\left(\sigma \cdot \left(p - \frac{e}{c}A\right)\right)^2}{2mc}v \Rightarrow$$

$$\frac{1}{2m}\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 v = (\mathcal{E}-e\varphi)v$$

Using $(\sigma\cdot B)(\sigma\cdot C) = B\cdot C + i\sigma\cdot(B\times C)$

with $B = C = \left(p-\frac{e}{c}A\right) \Rightarrow i\hbar\nabla$

$$\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 = \left(p-\frac{eA}{c}\right)^2 + i\sigma\left\{\left(p-\frac{e}{c}A\right)\times\left(p-\frac{e}{c}A\right)\right\} = \left(p-\frac{eA}{c}\right)^2 - \frac{e\hbar}{c}\sigma\cdot B$$

$p\times p + \frac{e^2}{c^2}A\times A + \underbrace{p\times A - A\times p}_{=2p\times A}$ $\quad p = i\hbar\nabla$

where $\bar{B} = \bar{\nabla}\times\bar{A}$

thus the eqn:

$$\frac{1}{2m}\left[\sigma\cdot\left(p-\frac{e}{c}A\right)\right]^2 v = (\mathcal{E}-e\varphi)v$$

becomes

$$\boxed{\frac{1}{2m}\left[\left(p-\frac{e}{c}A\right)^2 - \frac{e\hbar}{2mc}\sigma\cdot B + e\varphi\right]v = \mathcal{E}v}$$

$B = \nabla\times B$

PAULI EQUATION

the extra term $-\frac{e\hbar\sigma\cdot B}{2mc}$ suggest

that an electron in the magn field

gains extra energy $-\bar{\mu}\cdot\bar{B} = -\frac{e\hbar\sigma}{2mc}\cdot B = -\mu_B^{\sigma}\cdot B$

Important topic is spin-orbit interaction: READ pp 493-495 of the text.

the END OF RQM section!