HW 3 - Solutions

1. Plasma frequency

\[ m \ddot{x} = -eE \]
\[ x(t) = x e^{-i\omega t} \]
\[ -\omega^2 mx = -eE \]
\[ E(t) = E e^{-i\omega t} \]

\[ x = \frac{eE}{m\omega^2} \]

\[ p = -nex = -\frac{ne^2}{m\omega^2} E \]

\[ e(\omega) = 1 + 4\pi \frac{p(\omega)}{E(\omega)} \]

\[ E(\omega) = 1 - \frac{4\pi ne^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \]

\[ \omega_p = \sqrt{\frac{4\pi ne^2}{m}} \text{ (cgs)} \]

\[ \left( \text{N.B. } \omega_p = \frac{ne^2}{\sqrt{\varepsilon_0 m}} \text{ (SI)} \right) \]
Numerical estimate

\[ n \sim 10^{22} \text{ atoms/cm}^3 \]

\[ \varepsilon = 4.8 \times 10^{-10} \text{ eV/K} \]

\[ m \sim 9.1 \times 10^{-28} \text{ g} \]

\[ w_p \sim \sqrt{\frac{4\pi (10^{22}) (5 \times 10^{-10})^2}{9 \times 10^{-28}}} \]

\[ \sim \left( \frac{10}{3\sqrt{\pi}} \right) \times 10^{15} \]

\[ w_p \sim 6 \times 10^{15} \text{ a.u.}^{-1} \]
2. \( a) \quad E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 (3\pi^2 n)^{2/3}}{2m} \)

\[ \downarrow \]

\[ n = \left( \frac{2m E_F}{\hbar^2} \right)^{3/2} \frac{1}{3\pi^2} \]

\[ p = \frac{2n E_F}{5} \quad \Rightarrow \quad p = \frac{2}{15\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} E_F^{5/2} \]

\[ \frac{dp}{dE_F} = \frac{1}{3\pi^2} \left( \frac{2m E_F}{\hbar^2} \right)^{3/2} \]

\[ \Rightarrow \quad \frac{1}{3\pi^2} \hbar k_F^3 = n \]

\[ \downarrow \]

\[ \Delta p \approx n \Delta E_F \]

\( d) \quad n = 8.47 \times 10^{28} \text{ m}^{-3} \)

\[ \downarrow \]

\[ E_F \approx 1.1268 \times 10^{-18} \text{ J} \]

\[ \text{For } \Delta E_F = 10^{-6} E_F \]

\[ \Delta p \approx n \Delta E_F = 10^{-6} n E_F \]
\[ \Delta P \approx n \Delta E_F = 10^{-6} m E_F \]
\[ = 10^{-6} \times 8.47 \times 10^{2} \times 1.1268 \times 10^{-14} \]
\[ \approx 9.54 \times 10^{4} \frac{N}{m^2} \approx 0.94 \text{ atm} \]

(c) \[ g(E_F) = \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{1}{2\pi^2} E_F^{1/2} \]

\[ \Rightarrow \]
\[ \frac{dg(E_F)}{dE_F} = \frac{1}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \]
\[ \frac{1}{2E_F} \frac{1}{E_F} \left( \frac{2m}{\hbar^2} \right)^{3/2} \]
\[ = \frac{3n}{4E_F^2} \]

\[ \Delta g(E_F) \approx \frac{3n}{4E_F^2} \Delta E_F \]
\[ \Delta \varepsilon_F = 10^{-6} \varepsilon_F \]

\[ n = 8.47 \times 10^{28} \text{ m}^{-3} \]

\[ \varepsilon_F \approx 1.1268 \times 10^{-18} \text{ J} \]

\[ \downarrow \]

\[ \Delta g_\varepsilon(\varepsilon_F) \approx \frac{3n}{4 \varepsilon_f^2} \times 10^{-6} \varepsilon_F \]

\[ = \frac{3n}{4 \varepsilon_F} \cdot 10^{-6} \]

\[ \approx 5.64 \times 10^{-40} \text{ J}^{-1} \text{ m}^{-3} \]
3. Honeycomb Lattice

a) 

I find it easiest to choose the symmetric primitive vectors (1.3a) and (1.3b)

$$\vec{a}_1 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

$$\vec{a}_2 = a \left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

Then the basis vectors are

$$\vec{v}_1 = 0$$

$$\vec{v}_2 = a \left( 0, \frac{1}{\sqrt{3}} \right) = a \left( 0, \frac{\sqrt{3}}{3} \right)$$
Important distance

\[ | \frac{1}{v_1} - \frac{1}{v_2} | = \frac{a}{\sqrt{3}}. \]

\[ | \frac{1}{a_1} - \frac{1}{a_2} | = \frac{a}{\sqrt{3}} \sqrt{\left( \frac{\frac{3}{2} - \frac{1}{\sqrt{3}}} {2} \right)^2 + \left( \frac{1}{2} \right)^2} \]

\[ = \frac{3}{4} + \frac{1}{3} - 1 + \frac{1}{4} \]

\[ = a \sqrt{\frac{1}{3}} = \frac{a}{\sqrt{3}} \]

Similarly

\[ | \frac{1}{a_1} - \frac{1}{v_2} | = \frac{a}{\sqrt{3}} \]

Sides

All same length \( \frac{a}{\sqrt{3}} \)

b) \( a = 2.46 \ \text{Å} \implies d = \frac{2.46}{\sqrt{3}} = 1.42 \ \text{Å} \)
c) Density of graphene in gm/cm$^2$.

$$\rho = \frac{M_c}{\text{Area of u.c}} = \frac{2.12}{6.02 \times 10^{23}} \frac{g}{\sqrt{\frac{3}{2} a^2}}$$

Area of unit cell = $\frac{\sqrt{3}}{2} a^2$

$2.46 \text{ Å}$

$\rho = 7.6 \times 10^{-8} \text{ g/cm}^2$.
5. **Marder 1-4 Allowed Symmetries**

a) We start w/ Bravais lattice points (BLPs) at \( A = (0, 0) \) and \( B = (1, 0) \)

\[ \text{Rotations of } \pm \theta \]

\[ \downarrow \]

\[ C = (\cos \theta, \sin \theta) \quad \text{in a BLP} \]

\[ (1, 0) \quad \overset{\text{also}}{\longrightarrow} \quad (-1, 0) \]

BLP \quad BLP

Similarly

\[ D = (\cos \theta, \sin \theta) \quad \overset{\text{also}}{\longrightarrow} \quad E = (\cos \theta, -\sin \theta) \]

BLP

\[ \uparrow D - E \]

\[ (0, 2\sin \theta) \]
\( \Theta \)-rotational symmetry in Bravais lattice is only possible if

\[
\mathbf{r}_1 (1, 0) + \mathbf{r}_2 (\cos \Theta, \sin \Theta) = (0, 2 \sin \Theta)
\]

\( k \)

\( r_2 = 2 \)

\[
\mathbf{r}_1 + 2 \cos \Theta = 0
\]

\[
\cos \Theta = \frac{N}{\frac{2}{\pi}} \quad N \in \mathbb{Z}
\]

\[
\downarrow
\]

\[
\Theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

6-fold 4-fold 3-fold 2-fold
6. Let us represent the centers of the atoms (spheres with radius $r=a$) as points. Then a bottom layer hexagonal plaquette looks like

when we note that all the triangles are equilateral

$$h_e p = A B A B \ldots$$

The spheres of the second layer of an hcp structure are placed so that each is equidistant from the three neighboring triangular vertices (associated with the A layer).

Therefore a sphere on the second layer would be placed so that its center is at

$$\left( \frac{a}{\sqrt{3}}, 0, \frac{c}{2} \right)$$

directly over

$$\left( \frac{a}{\sqrt{3}}, 0, \frac{c}{2} \right)$$. The centers of
In order to solve for $c$, we study the right triangle.

\[
\frac{c}{2} + \frac{a^2}{3} = a^2
\]

\[
\left(\frac{c}{2}\right)^2 = \frac{2}{3} a^2
\]

\[
\left(\frac{c}{a}\right)^2 = \frac{8}{3} \quad \Rightarrow \quad \frac{c}{a} = \sqrt{\frac{8}{3}}
\]

\[
\frac{c}{a} = 1.633.
\]
7. In order to determine the \(x-y\)-intercepts of these planes, we must invert the indices (thus running the Miller index prescription backwards):

\[
\begin{align*}
(0 1) & \Rightarrow (0 \overline{1}) \\
(1 2) & \Rightarrow (1 \frac{1}{2}) \Rightarrow (2 1) \\
(2 3) & \Rightarrow (\frac{1}{2} \frac{1}{3}) \Rightarrow (3 2) \\
(1 \overline{2}) & \Rightarrow (1 \frac{1}{2}) \Rightarrow (2 1)
\end{align*}
\]

The planes are displayed for a rectangular lattice \((a_2 = 2a_1)\) on the next page.

\[\frac{a_2}{a_1} = 2\]
8. Quasicrystals are solids whose constituent atoms are arranged in neither periodic nor random structure. Diffraction from quasicrystals yields sharp spots despite the absence of translational symmetry. The constituents of quasicrystals are arranged in non-repeating structures that obey specific tiling rules.

9. The diffraction patterns of quasicrystalline materials displayed underlying "forbidden symmetries" that are known to be incompatible with space-filling structures. At the time of the discovery of these quasicrystalline materials it was believed that there exist two types of solid: a (periodic) crystal and a (random) glass. Furthermore, it was believed that the presence of sharp spots in a diffraction pattern was indicative of underlying periodic order, which would not occur for structures with "forbidden" symmetries. Finally, it was not clear
how a quasi-regular quasicrystal could develop using "local rules" but this was shown mathematically after a few years.

10. Diffraction Patterns of a Crystal and a Quasicrystal

(a) Similarities

- Sharp spots

Patterns indicate some similar rotational symmetries (2- and 4-fold)

(b) Differences.

Quasicrystalline patterns display rotational symmetries (e.g. 5-fold) forbidden in crystalline materials

11. (a) Conductivity

The conductivity of a quasicrystalline material is less than its crystalline counterpart

(b) Hardness

A quasicrystal is harder to deform than its crystalline counterpart
12. Two possible applications of quasicrystals

(a) Low-friction coatings
    - Brochure, engine components

(b) Hardness
    - Possible replacement in industrial diamonds

13. In principle an icosahedral glass should have
    broader diffraction peaks than a quasicrystal
    since the latter but not the former has prismatic
    order. However, practically there are always random
    distortions that develop during growth, so that
    most quasicrystalline materials are not ideal.
    However, signatures of stress-induced phase
    strain, a distortion of the structure obtained
    by rearranging cells, will distinguish them; in
    the icosahedral glass it will be isotropic whereas
    it will be highly anisotropic for the quasicrystal.