Physics 601, Fall 2016

Final Exam
December 16, 2016

Name [Solutions]

The ten problems are worth 10 points each.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
1. **Effects of hydrostatic pressure.** When hydrostatic pressure is supplied to a three-dimensional metal, its Fermi energy increases because the electron density increases.

(a) Derive an expression for the change in the Fermi energy within the free electron model.

\[ \varepsilon_F = \frac{k^2 k_F^2}{2m} = \frac{k^2 (3\pi^2 n)^{2/3}}{2m} \implies n = \left( \frac{2m \varepsilon_F}{k^2} \right)^{3/2} \frac{1}{3\pi^2} \]

\[ P = \frac{2n \varepsilon_F}{5} \implies P = \frac{2}{15\pi^2} \left( \frac{2m}{k^2} \right)^{3/2} \varepsilon_F^{5/2} \]

\[ \frac{dP}{d\varepsilon_F} = \frac{1}{3\pi^2} \left( \frac{2m \varepsilon_F}{k^2} \right)^{3/2} = \frac{1}{3\pi^2} v k_F \frac{3}{3} = n \]

\[ \Delta P \approx n \Delta \varepsilon_F \]  

(b) Compute the pressure that is required to change the Fermi energy for a factor of 1.000001 for Cu. The electron density in Cu is \(8.47 \times 10^2 \) m\(^{-3}\).

\[ n = 8.47 \times 10^2 \text{ m}^{-3} \]

\[ \varepsilon_F \approx 1.13 \times 10^{-18} \text{ J} \]

\[ \Delta \varepsilon_F = 10^{-6} \varepsilon_F \]

\[ \Delta P \approx n \Delta \varepsilon_F = 10^{-6} n \varepsilon_F \]

\[ = (10^{-6}) (8.47 \times 10^{28}) (1.13 \times 10^{-18}) \]

\[ \approx 9.57 \times 10^4 \text{ N/m}^2 = 90 \text{ kPa} \]

\[ (1 \text{ Pa} = 1 \text{ N/m}^2) \]
(c) Evaluate the change in the density of states at the Fermi surface for Cu when the Fermi energy is changed by a factor of 1.000001.

\[ g(e_F) = \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{1}{2\pi^2} e_F^{1/2} \]

\[ \frac{dg(e_F)}{de_F} = \frac{1}{4\pi^2 e_F^{1/2}} \left( \frac{2m}{\hbar^2} \right)^{3/2} \]

\[ = \frac{1}{4\pi^2 e_F^2} \left( \frac{2m e_F}{\hbar^2} \right)^{3/2} \]

\[ = \frac{3n}{4 e_F^2} \]

\[ \Delta g(e_F) = \frac{3n}{4 e_F^2} \Delta e_F \]

\[ \Delta g(e_F) = 5.64 \times 10^{40} \text{ J}^{-1} \text{ m}^{-3} \]
2. a) Using the free electron model, and denoting the electron concentration by \( n \), show that the radius of the Fermi sphere in \( k \)-space is given by

\[
k = \left(\frac{3\pi^2 n}{\lambda^2}\right)^{\frac{1}{3}}
\]

\[
\mathcal{N} = \frac{1}{(2\pi)^3} \frac{4}{3} \pi k_F^3 \\
\downarrow
\]

\[
\mathcal{N} = \frac{1}{2\pi^2} k_F^3 \Rightarrow k_F = \left(\frac{3\pi^2 n}{\lambda^2}\right)^{\frac{1}{3}}
\]

b) As the electron concentration increases, the Fermi surface expands. Show that this sphere touches the faces of the first zone in an fcc lattice when the electron-to-atom ratio \( n/n_{\lambda} = 1.36 \) where \( n_{\lambda} \) is the atom concentration.

Fermi surface touches the face of the 1st BZ of the fcc lattice when

\[ k_F = k_i \]

Reciprocal lattice vectors for fcc

\[
a^* = \frac{2\pi}{a} (1, -1, 1) \quad c^* = \frac{2\pi}{a} (-1, 1, 1)
\]

\[
\begin{align*}
\mathbf{a}^* &= \frac{2\pi}{a} (1, 1, -1) \\
\lambda_i &= \frac{1}{\sqrt{3}} \frac{\pi}{a} \\
\end{align*}
\]
\[ \tilde{n} = \frac{4 \text{ atoms}}{a^3}, \quad \frac{n}{n_a} \uparrow \equiv \text{electrons/atom} \]

\[ k_F = k_i \]

\[ \left( \frac{3\pi^2}{4} \frac{\frac{\pi}{n_a}}{a^3} \right)^{1/3} = \sqrt{3} \frac{\pi}{a} \]

\[ \left( \frac{12}{\pi} \right)^{1/3} \left( \frac{n}{n_a} \right)^{1/3} = \sqrt{3} \]

\[ \frac{n}{n_a} = \sqrt{3} \frac{\pi}{\sqrt{12} 4} = \frac{\sqrt{3} \pi}{4} = 1.36. \]
c) Let's suppose that some of the atoms in a Cu crystal, which has an fcc lattice, are gradually replaced by Zn atoms. Considering that Zn is divalent while Cu is monovalent, calculate the atomic ratio of Zn to Cu in a CuZn alloy (brass) in which the Fermi surface touches the zone faces. Use the free-electron model. (This particular mixture is interesting because the solid undergoes a structural phase change at this concentration ratio).

\[ n = \frac{4}{a^3} \left\{ (1-x)(1) + x(e) \right\} = \frac{4}{a^3} \left\{ 1 + x \right\} \]

\[ \frac{k_F}{k_i} = \Rightarrow \left( \frac{3\pi^2 n}{\pi} \right)^{1/3} = \sqrt{3} \frac{\pi}{a} \]

\[ \left( \frac{12}{\pi} \right)^{1/3} (1+\chi)^{1/3} = \sqrt{3} \Rightarrow (1+\chi) = \frac{\pi}{2} \cdot 3 \sqrt{3} \]

\[ \chi = \frac{\pi \sqrt{3}}{4} - 1 = 0.36 \]

**Atomic ratios**

\[ X = \frac{x_{Zn}}{A_{Zn}} = \frac{36}{65.4} = 0.55 \]

\[ Y = \frac{x_{Cu}}{A_{Cu}} = \frac{64}{63.5} = 1.007 \]

\[
\text{divide each by 0.55} \quad \Rightarrow \quad \text{Zn}_{1.007} \text{Cu}_{1.832} \quad \Rightarrow \quad \text{Zn}_{5} \text{Cu}_{9}
\]

Formula unit should have integer # of constituent atoms.
3. a) Experimentally what is the difference between a metal and an insulator?

Application of an electric field to a metal results in current flow.

b) What is the criterion, based on energy-band theory, for distinguishing a metal from an insulator?

A solid behaves as a metal if some of its bands are partially occupied. Filled bands carry no currents; insulators must have even #s of electrons per unit cell.

c) What is a semiconductor and how is it distinctive from a metal and from an insulator?

A semiconductor has a gap \( \Delta < 2 \text{ eV} \) so that electrons can be thermally excited into the conduction band at operating temperatures. Therefore, unlike an insulator, a semiconductor is conducting at high temperatures (\( T \sim 350 \text{ K} \)); however, its conductivity is substantially lower than that of a metal due to its relatively low carrier concentration.
d) What is an intrinsic semiconductor?

An intrinsic semiconductor has a carrier concentration determined by thermally-induced interband transitions; here n = p.

e) What is an extrinsic semiconductor?

In an extrinsic semiconductor, impurities are introduced to add holes or electrons to the system; these then become the principal charge carriers.

f) Please sketch a n(T) vs. T plot (temperature-dependence of the carrier concentration) for an n-type (electron-doped) semiconductor indicating the three main temperature regimes with brief descriptions of each.

\[ n \]

\[ T \]

\[ T_{1} \]

\[ T_{2} \]

T < T₁, 
all electrons are "frozen out" at impurity sites

T₁ < T < T₂, 
extrinsic regime, n ~ N_d

T₂ < T, 
thermal activation of e's from VB → CB dominant processes.
4. a) Please explain why the Meissner effect indicates that a superconductor is not a perfect conductor.

Meissner Effect

\[ T > T_c \]

\[ T < T_c \]

Perfect conductor

\[ j = \sigma E \]

\[ E = \oint j_{\text{finite}} \Rightarrow E = 0 \]

Maxwell's Equations

\[ \nabla \times E = \frac{\partial B}{\partial t} \Rightarrow \nabla \times E = 0 \]

Supercurrent \Rightarrow \text{thermodynamic not dynamic diamagnet}!

\[ j \propto A \Rightarrow B = B_0 e^{-x/\lambda} \]
b) Use the Heisenberg uncertainty principle to estimate the coherence length of a superconductor.

\[ \Delta x \Delta p \sim h \Rightarrow \Delta x \sim \frac{h}{\Delta p} \]

Only electrons within of \( kT_c \)
of the Fermi level can participate in the development of superconductivity at \( T \sim T_c \).

\[ \Delta p \sim \left( \frac{kT_c}{e_F} \right) (m v_F) \sim \frac{kT_c}{v_F} \]

\[ \Delta x \sim \frac{h v_F}{kT_c} \Rightarrow \xi \sim \frac{h v_F}{kT_c} \]

\[ \xi \sim \frac{k v_F}{kT_c} \sim \frac{h v_F}{kT_c} \]
c) Please describe the difference between Type I and Type II superconductors using H-T phase diagrams and length-scales.

\[ H \]

\[ H_c \]

\[ T_c \]

\[ T \]

\[ \xi > \lambda \rightarrow \text{Bodhlin penetration length} \]

\[ \xi < \lambda \]

\[ \text{Type I} \]

\[ \text{Type II} \]

\[ \text{Vortex} \]

\[ \text{SC} \]

\[ \text{N} \]

\[ \text{T} \]

\[ \text{d) Why do we expect a high-temperature superconductor to be Type II?} \]

In d) we found \( \xi \sim \frac{1}{T_c} \)

so that if \( T_c \) large, \( \xi \) is small.

Since a Type II SC occurs when \( \xi < \lambda \), we expect a small \( \xi \) SC to be Type II.
5. Why are metals shiny?

For normal incidence (simplest case) of a plane wave, the reflectivity is

\[ R = \left| \frac{1 - N}{1 + N} \right|^2 \]

where \( N = \sqrt{\varepsilon} = n + ik \)

In the free electron model (\( \omega T >> 1 \) (Visible/UV))

\[ \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \]

\( \omega < \omega_p \Rightarrow \varepsilon < 0 \Rightarrow n = 0 \)

\[ \downarrow \]

\[ R = 1 \]

Perfect reflectivity \( \Rightarrow \) shining!

The erins of metals are due to intraband transitions that are beyond the free electron (classical) approach.
6. Density of States for Quantum Wells

a) Consider a quantum well as shown in the figure below. Calculate the density of states for electrons and holes in the quantum well. Hint: it is a 2D electron gas, but don't forget that there are several particle-in-a-box states.

For 2d \( g(e) = \frac{mA}{\hbar^2 \pi} \quad A = \text{area} \)

Density of states/area = \( g_A(e) = \frac{m}{\hbar^2 \pi} \)

These density of states in \((x, y)\) accompany states with each value of \( k_x, k_y \) in \( (n, n_x) \).

Therefore, the density of states/area of the QW is

\[ g_A(e) = \frac{m^*}{\hbar^2} \sum_{n_z} \Theta(E - E_{n_z}) \]

where \( E_{n_z} = \frac{\hbar^2 n_z^2}{2 m^* L^2} = " \text{envelope energy in the } Z \text{-direction} \) and \( m^* \) is the effective mass (only difference between e's and holes).
b) Consider a so-called "quantum wire" which is a one-dimensional wire of GaAs embedded in surrounding AlGaAs (You can consider the wire cross-section to be a square box). Describe the density of states for electrons or holes within the quantum wire. Why might this quantum wire make a very good laser?

\[ g(e) = \frac{1}{\frac{dE}{dk}} \frac{dN}{dk} \frac{1}{dE} \]

1d \[ \bar{N}(k) = \frac{k}{\frac{1}{2\pi/L}} \Rightarrow \frac{dN}{dk} = \frac{L}{2\pi} \]

\[ \frac{dE}{dk} = \frac{\hbar^2 k}{2m^*} = \frac{\hbar^2}{2m^*} \sqrt{\frac{2m^*E}{\hbar^2}} = \frac{\sqrt{2\hbar^2/\sqrt{m^*E}}} \]

\[ g(E) = \frac{L}{\pi} \sqrt{\frac{m^*}{2\hbar^2}} \frac{1}{\sqrt{E}} \Rightarrow g_L(E) = g(E) \]

The energy states are confined in the \( \frac{1}{L} \) unit length in the \( x, y \) directions, while \( m^* \) will be different for e's and h's. The highly discretized nature of \( g_L(E) \) makes for excellent LEDs and laser nanowires.
7. a) What is the n=1 Bragg angle for reflection from the (110) planes in bcc iron (cube edge = 2.91 Å) for an X-ray wavelength of λ = 1.54 Å?

Bragg's Law
\[ \lambda = 2d \sin \theta \]
\[ \sin \theta = \frac{\lambda}{2d} \]

bcc lattice
\[ d = \frac{a}{\sqrt{h^2 + k^2 + l}} \]
\[ d_{110} = \frac{a}{\sqrt{2}} \]

\[ \sin \theta = \frac{\lambda}{2d} \]
\[ = \frac{(1.54)}{(2.91)} \frac{\sqrt{2}}{2} = 0.374 \]

\[ \theta = 21.96^\circ \Rightarrow \theta \approx 22^\circ \]
b) Please calculate the density of bcc iron. You can take the atomic weight of Fe to be 56 and you may need Avogadro’s number $N = 6.02 \times 10^{23} \text{ mol}^{-1}$.

$$\rho = \frac{\# \text{ Fe atoms}}{\text{Volume of unit cell}} \cdot \frac{\text{Molecular weight of Fe (g-atom)}}{N (\# \text{ atoms in 1 g-mol})}$$

$$= \frac{2}{(2.91 \times 10^{-8})^3 \text{ cm}^3} \cdot \frac{56 \text{ g}}{6.02 \times 10^{23} \text{ atoms}}$$

$$= \frac{(2)(56)}{(2.91)^3 (6)} \frac{10^{1}}{\text{g/cm}^3}$$

$$= 7.58 \text{ g/cm}^3$$

N.B. Textbook value

$$\rho = 7.86 \text{ g/cm}^3$$

for Fe!
8. a) Please draw an energy band diagram for a p-n junction (in thermal equilibrium).

b) Please sketch the current-voltage curve of a p-n junction (for V positive and negative)
c) Please describe a tunnel diode and how it works.

- Usual diode \( \rightarrow \) speed limited by diffusion
- TD = tunnel diode \( \rightarrow \) QM tunnelling

\[ \downarrow \]

-q Typ. switching

- Heavily doped pn junction w/ depletion

\[ \text{region } \leftrightarrow \text{no bias} \]

\[ \text{equilibrium} \]

\[ \text{small forward bias} \]

\[ \text{tunnel diode} \]

\[ \frac{\Delta V}{\Delta i} < 0 \]

\[ \text{ordinary pn junction} \]

\[ \text{small voltage change} \rightarrow \text{large current change} \]
d) What is negative differential resistance and what is an important application of it?

\[
\frac{\Delta v}{\Delta i} < 0
\]

small voltage change \[\downarrow\] large \( \Delta i \) at high speed

\[
R_{\text{diff}} = \frac{dv}{di} < 0
\]

for given voltage range

AC and DC currents flow in opposite directions (must have)

Sviva can converter DC \( \rightarrow \) AC power \[\uparrow\] \( \Rightarrow \)

Negative (differential) resistance \( \Rightarrow \)

hysteresis (switching)

substantial oscillation

(reproducible, reversible, nonlinear (slow) oscillation)
9. The diagram below shows the band structure for some hypothetical 1D solid. At T=0 will this solid be a metal or an insulator if it has:

a) 1 electron/primitive cell?
b) 2 electrons/primitive cell?
c) 3 electrons/primitive cell?
d) 4 electrons/primitive cell?

In a paragraph with a minimum of four sentences, please justify your answers including the important concepts involved in your reasoning.
Key concepts:

1. $3N$ states per band
2. $x$ electrons per cell $\Rightarrow Nx$ electrons
3. A band that is completely full carries no current / a solid behaves as a metal when some of its bands are partially occupied.

$\text{1 } e/\text{cell} \Rightarrow N \text{ electrons } \Rightarrow 1\text{st band}$

$\frac{1}{2}$ full

$\text{2 } e/\text{cell} \Rightarrow 2N \text{ electrons } \Rightarrow 1\text{st band}$

full

$\text{3 } e/\text{cell} \Rightarrow 3N \text{ electrons } \Rightarrow 2\text{nd band}$

$\frac{1}{2}$ full

Metal

$\text{4 } e/\text{cell} \Rightarrow 4N \text{ electrons } \Rightarrow 2\text{nd band}$

full

Insulators.
10. In a minimum of six sentences please give a “pitch” for nanomaterials in transistors giving a flavor for the promise and the challenges of these molecular devices.

**Pitch for their promise**

- Small size and low resistance (hence low power)
- Superb charge transport properties (high mobility, tunable band gap)
- Low capacitance → potential for reduced power dissipation compared to conventional circuitry
- Electrochemiluminescence → use in photonics / optoelectronics
- Substrate independence

**Challenges**

- Large-scale reliable fabrication w/ specified parameters
- Quality of metal/nanomaterial interfaces crucial for contacts
- Controlled synthesis and placement